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$\pi\eta$ scattering and the resummation of vacuum fluctuation in three-flavor χPT

M. Kolesár[a], J. Novotný

Institute of Particle and Nuclear Physics, Charles University, V. Holešovičkách 2, 18000, Prague 8, Czech Republic

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Abstract. We discuss various aspects of resummed chiral perturbation theory, which was developed recently in order to consistently include the possibility of large vacuum fluctuations of the $\overline{s}s$ pairs and the scenario with smaller value of the $\overline{q}q$ condensate for $N_f = 3$. The subtleties of this approach are illustrated using a concrete example of observables connected with $\pi\eta$ scattering. This process seems to be a suitable theoretical laboratory for this purpose due to its sensitivity to the values of the $O(p^4)$ LECs, namely to the values of the fluctuation parameters L_4 and L_6. We discuss several issues in detail, namely the choice of "good" observables and properties of their bare expansions, the "safe" reparametrization in terms of physical observables, the implementation of exact perturbative unitarity and exact renormalization scale independence, the role of higher order remainders and estimates of their influence. We make a detailed comparison with standard chiral perturbation theory and use generalized χPT as well as resonance chiral theory to estimate the higher order remainders.

1 Introduction

As is well known, at the energy scales $E \ll \Lambda_H \sim 1\,\text{GeV}$ the physics of QCD is non-perturbative and is governed by chiral symmetry (χS) $\text{SU}(N_f)_\text{L} \times \text{SU}(N_f)_\text{R}$. This global symmetry is present on the classical level within QCD with N_f massless quarks (in the chiral limit of QCD), and on the quantum level there exist strong theoretical (for $N_f \geq 3$) and phenomenological arguments for spontaneous symmetry breakdown (SSB) of χS according to the pattern $\text{SU}(N_f)_\text{L} \times \text{SU}(N_f)_\text{R} \to \text{SU}(N_f)_V$. Due to confinement, quark and gluon fields do not represent appropriate low energy degrees of freedom within the above mentioned energy range; the relevant degrees of freedom correspond to the lightest colorless hadrons in the QCD spectrum. As far as the Green functions of quark currents are concerned, it is possible to obtain a general solution of the chiral Ward identities in terms of the low energy expansion. This expansion can be organized most efficiently using the methods of effective field theory corresponding to the low energy limit of QCD with N_f light quark flavors, which is known as chiral perturbation theory (χPT) [1–3]. χPT describes the low energy QCD dynamics in terms of the lightest (N_f^2-1)-plet of the pseudoscalar mesons identified with the Goldstone bosons (GB) of the spontaneously broken chiral symmetry, which appear in the particle spectrum of the theory as a consequence of the Goldstone theorem. In the chiral limit these pseudoscalars are massless and dominate the low energy dynamics of QCD. They interact weakly at low energies $E \ll \Lambda_H$, where $\Lambda_H \sim 1\,\text{GeV}$ is the hadronic scale corresponding to the masses of the lightest non-Goldstone hadrons. This feature of the GB dynamics enables a systematic perturbative treatment with the expansion parameter (E/Λ_H). Within real QCD, the quark mass term $\mathcal{L}_{f,\text{mass}}^{\text{QCD}}$ breaks χS explicitly and the Goldstone bosons become pseudo-Goldstone bosons (PGB) with nonzero masses. Though $m_f \neq 0$, for $m_f \ll \Lambda_H$ the mass term $\mathcal{L}_{f\text{mass}}^{\text{QCD}}$ can be treated as a perturbation. As a consequence, PGBs correspond to the lightest hadrons in the QCD spectrum[1] (identified with $\pi^0, \pi^\pm$ for $N_f = 2$ and $\pi^0, \pi^\pm, K^0, \overline{K^0}, K^\pm, \eta$ for $N_f = 3$) and the interaction of PGB at the energy scale $E \ll \Lambda_H$ continues to be weak. Because $M_P < \Lambda_H$, the QCD dynamics at $E \ll \Lambda_H$ is still dominated by these particles and the effective theory provides us with a simultaneous expansion in powers of (E/Λ_H) and (m_f/Λ_H). The Lagrangian of χPT can be constructed on the basis of symmetry arguments only; the unknown information about the nonperturbative properties of QCD are hidden in the parameters known as low energy constants (LEC) [2, 3]. These are related to the (generally nonlocal) order parameters of the SSB of χS, the most prominent of them being the Goldstone boson decay constant F_0 and the chiral condensate[2] $B_0 = \Sigma/F_0^2$, where $\Sigma = -\langle\overline{u}u\rangle_0$.

[1] The PGB masses M_P can be expanded in the powers (and logarithms) of the quark masses starting from the linear term and therefore vanish in the chiral limit.

[2] The parameter F_0 is, however, more fundamental in the sense that $F_0 \neq 0$ is both a necessary and sufficient condition for SSB, while $\langle\overline{q_f}q_f\rangle_0 \neq 0$ corresponds to the sufficiency condition only. (The lower index zero here means the chiral limit.)

[a] e-mail: kolesar@ipnp.troja.mff.cuni.cz

To be more precise, N_f-flavor χPT is in fact an expansion in m_i, around the $\mathrm{SU}(N_f)_\mathrm{L} \times \mathrm{SU}(N_f)_\mathrm{R}$ chiral limit $m_i = 0$, $i \leq N_f$, while keeping all the other quark masses for $i > N_f$ at their physical values. Because the $m_{u,d}$ are much smaller not only in comparison with the hadronic scale Λ_H but also in comparison with the intrinsic QCD scale Λ_QCD, the two-flavor χPT is expected to produce a well-behaved expansion corresponding to small corrections to the $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$ chiral limit.

The strange quark mass, on the other hand, though still small enough with respect to Λ_H to be treated as an expansion parameter within three-flavor χPT (relating real QCD with its $\mathrm{SU}(3)_\mathrm{L} \times \mathrm{SU}(3)_\mathrm{R}$ chiral limit), is of comparable size with respect to Λ_QCD. This fact, besides the expected worse convergence of the three-flavor χPT, might also have interesting consequences for the possible difference between the $N_f = 2$ and $N_f = 3$ chiral dynamics. As discussed intensively in a series of papers [4–9], $m_s \lesssim \Lambda_\mathrm{QCD}$ suggests that the loop effects of the vacuum $\overline{s}s$ pairs are not suppressed as strongly as for the heavy quarks and might enhance the magnitude of the $N_f = 2$ chiral order parameters relatively to their $N_f = 3$ chiral limits. This applies mainly to $F_0(N_f)$ and $\Sigma(N_f) = F_0^2(N_f)B_0(N_f)$, which should satisfy the paramagnetic inequalities [4]

$$\begin{aligned} \Sigma(2) &> \Sigma(3) = \lim_{m_s \to 0} \Sigma(2)\,, \\ F_0(2) &> F_0(3) = \lim_{m_s \to 0} F_0(2)\,. \end{aligned} \tag{1}$$

The leading order difference between the two-flavor and three-flavor values is proportional to m_s, with coefficients measuring the violation of the OZI rule in the 0^{++} channel [4]; e.g.

$$\Sigma(2) = \Sigma(3) + m_s \overline{Z}_1^s + \dots\,, \tag{2}$$

where

$$\overline{Z}_1^s = \lim_{m_s \to 0} \int \mathrm{d}^4x \langle \overline{u}u(x)\overline{s}s(0)\rangle_c \tag{3}$$

and analogously for F_0. The fluctuation parameter $\overline{Z}_1^s$ is related to the LEC $L_6^r(\mu)$ (and $L_4^r(\mu)$ for F_0) of three-flavor χPT. As discussed in [4–9], these parameters might be larger than their estimate based on the large N_c expansion, provided $N_f = 3$ is close to the critical number of light quark flavors N_f^crit for which the chiral symmetry is restored. Available estimates vary widely, some indicating a larger number, $N_f^\mathrm{crit} \sim 10$–$12$, for $N_c = 3$ [10–12], while other approaches [13, 14] and lattice calculations [15–19] discuss a possibly much lower value, $N_f^\mathrm{crit} \leq 6$. Provided the scenario of large vacuum fluctuations is relevant, the second term in (2) (called the induced condensate in [5, 8, 20]) may be numerically comparable with the first term and the three-flavor condensate $\Sigma(3)$ could be substantially smaller than the two-flavor one, the value of which is experimentally accessible in recent experiments. Analogous reasonings apply to the relationship of $F_0(2)$ and $F_0(3)$.

These effects could possibly have strong consequences for the organization of the chiral expansion in the $N_f = 3$ case [4, 6–8]. Let us recall that the general form of the Lagrangian of χPT is

$$\mathcal{L} = \sum_{m,n} \mathcal{L}^{(m,n)}\,, \tag{4}$$

where

$$\mathcal{L}^{(m,n)} = \sum_k C_k^{(m,n)} O_k^{(m,n)}\,, \tag{5}$$

with the LECs $C_k^{(m,n)}$ and the independent set of the operators $O_k^{(m,n)} = O(\partial^m m_f^n)$.

In order to be able to treat the double expansion consistently, it is necessary to assign a single integer parameter called the *chiral order* to each term $\mathcal{L}^{(m,n)} = O(\partial^m m_f^n)$ of the effective Lagrangian. The terms $\mathcal{L}_k$ with chiral order k are then called $O(p^k)$ terms. Obviously, $\partial = O(p)$. A matter of discussion might be, however, the question of the chiral power of m_f. This question is intimately connected to the scenario according to which the SSB of χS is realized.

The standard scenario [2, 3] corresponds to the assumption that the SSB order parameters $\Sigma(N_f)$ and $F_0(N_f)$ are large in the sense that the ratios

$$X(N_f) = \frac{2\widehat{m}\Sigma(N_f)}{F_\pi^2 M_\pi^2}\,, \tag{6}$$

(where $\widehat{m} = (m_u + m_d)/2$) and

$$Z(N_f) = \frac{F_0^2(N_f)}{F_\pi^2} \tag{7}$$

are close to 1. Because $M_\pi^2 = O(p^2)$, it is then natural to take $m_f = O(p^2)$, i.e. $k = m + 2n$. This results in the *standard* χPT (SχPT in what follows). This scenario seems to be experimentally confirmed [21] for $N_f = 2$; a recent analysis of the data yields [22]

$$X(2) = 0.81 \pm 0.07\,, \quad Z(2) = 0.89 \pm 0.03\,. \tag{8}$$

The $O(p^2)$ Lagrangian [2, 3]

$$\mathcal{L}_2 = \frac{F_0^2}{4}\left(\langle \partial_\mu U^+ \partial^\mu U\rangle + 2B_0\langle U^+\mathcal{M} + \mathcal{M}^+ U\rangle\right) \tag{9}$$

gives $\Sigma(2)_\mathrm{LO} = \Sigma(3) = B_0 F_0^2$ at the leading order, thus postponing the difference $\Sigma(2) - \Sigma(3)$ to higher orders. The same is true for the parameters $F_0(N_f)$. Let us also note that the quark mass ratio $r = m_s/\widehat{m}$ is not a free parameter here[3]. At the leading order one has

$$r = 2\frac{M_K^2}{M_\pi^2} - 1\,. \tag{10}$$

An alternative way of chiral power counting for $N_f = 3$ is the *generalized* χPT (GχPT) [24–29], originally de-

[3] On the contrary, the value of r is usually taken as input in standard $O(p^6)$ fits; see e.g. [23] and references therein.

signed to treat the scenario with small quark condensate $X(3) \ll 1$ and to take the quark mass ratio r as a free parameter. In the case $X(3) \ll 1$ it is natural to take $m_f = O(p)$ and $B_0 = O(p)$; this means that $k = m + n$. In contrast to $S\chi$PT, there are also odd chiral orders and the $O(p^2)$ Lagrangian contains additional terms, which are $O(p^4)$ within the standard chiral counting[4] (see e.g. [24, 25, 28]):

$$\mathcal{L}_2 = \frac{F_0^2}{4}\big(\langle\partial_\mu U^+\partial^\mu U\rangle + 2B_0\langle U^+\mathcal{M} + \mathcal{M}^+U\rangle + A_0\langle(U^+\mathcal{M})^2 + (\mathcal{M}^+U)^2\rangle + Z_0^P\langle U^+\mathcal{M} - \mathcal{M}^+U\rangle^2 + Z_0^S\langle U^+\mathcal{M} + \mathcal{M}^+U\rangle^2\big)\,. \quad (11)$$

For the condensate $\Sigma = -\langle\overline{u}u\rangle$ we get at the leading order for $N_f = 3$

$$\Sigma_{\mathrm{LO}} = B_0F_0^2 + Z_0^S(2\widehat{m} + m_s) = \Sigma(3) + Z_0^S(2\widehat{m} + m_s) \quad (12)$$

and therefore

$$\Sigma_{\mathrm{LO}}(2) = \Sigma(3) + Z_0^S m_s\,. \quad (13)$$

This allows the difference $\Sigma(2) - \Sigma(3)$ to appear already at the leading order, consistently with the small $\Sigma(3)$ scenario. The next-to-leading order $O(p^3)$ Lagrangian

$$\mathcal{L}_3 = \frac{F_0^2}{4}\big(\xi\langle\partial_\mu U^+\partial^\mu U U^+\mathcal{M} + \mathcal{M}^+U\rangle + \tilde{\xi}\langle\partial_\mu U^+\partial^\mu U\rangle\langle U^+\mathcal{M} + \mathcal{M}^+U\rangle + \ldots\big) \quad (14)$$

(where the ellipsis stands for the additional terms, which are of the order $O(p^6)$ in SχPT) gives rise to the $N_f = 3$ relation

$$F^2_{\pi,\mathrm{NLO}} = F_0^2(3)(1 + 2\tilde{\xi}(m_s + 2\widehat{m}) + 2\widehat{m}\xi)\,, \quad (15)$$

which implies that the difference $F_0^2(2) - F_0^2(3)$ is treated as an effect of the next-to-leading order

$$F_0^2(2) = F_0^2(3)(1 + 2\tilde{\xi}m_s)\,. \quad (16)$$

Therefore, neither SχPT nor GχPT may encompass the case of large fluctuation parameter $\tilde{\xi}$, and the ratio $Z(3) \ll 1$ at the leading order.

Quite recently, a consistent method of handling the case $X(3), Z(3) \ll 1$ was proposed [4, 6–9]. Instead of changing the chiral power counting, it is based on more careful manipulations with the chiral expansion. As was discussed in the above references, the case $X(3)$, $Z(3) \ll 1$ could significantly influence the properties of the chiral expansion inducing instabilities of the perturbative series corresponding to the observables, which cannot be linearly related to the QCD correlators (such as the *ratios* like PGB masses, scattering amplitudes etc.). For such quantities, one should not perform a perturbative chiral expansion of the denominators but rather keep the ratios in a non-perturbative "resummed" form. The possibly large vacuum $\overline{s}s$ pair fluctuations are then parameterized in terms of $X(3)$, $Z(3)$ and r and treated as free parameters. We return to a detailed formulation of this recipe in the next section.

The aim of this paper is to illustrate the "resummed" form of the chiral expansion with special attention to its formal properties and to the details and subtleties of the general procedure. Motivated by our preliminary results on $\pi\eta$ scattering within GχPT [34], we have chosen the observables connected with this process as a concrete example which seems to be sensitive to the deviations from the standard assumption $X(3), Z(3) \sim 1$ (note that some recent phenomenological studies suggest the possibility of $X(3) \sim 0.5$, cf. [6, 30–33]). Also, from the phenomenological point of view, the off-shell $\pi\eta\pi\eta^*$ vertex is a necessary building block for the non-resonant part of the amplitude for the rare decay $\eta \to \pi^0\pi^0\gamma\gamma$. Preliminary estimates within GχPT [35, 36] suggest that the effect of a deviation of this off-shell vertex from the standard case might, at least in principle, be observed. The details will be presented elsewhere [37].

The amplitude of $\pi\eta$ scattering was already calculated within SχPT to $O(p^4)$ (and within extended SχPT with explicit resonance fields) in [38], where the authors presented a prediction for the scattering lengths and phase shifts of the S, P and D partial waves. We here quote their $O(p^4)$ results for the S- and P-wave scattering lengths (in units of the pion Compton wavelength): $a_0^{\mathrm{S}\chi\mathrm{PT}} = 7.2 \times 10^{-3}$ and $a_1^{\mathrm{S}\chi\mathrm{PT}} = -5.2 \times 10^{-4}$.

The paper is organized as follows. In Sect. 2 we recapitulate the motivation for the resummed version of χPT and the construction of the bare expansion of "good" observables. We make a detailed general discussion, connected with the four-meson amplitude, of the strict chiral expansion, the dispersive representation and the matching of both approaches, stressing the reconciliation of exact perturbative unitarity and the exact renormalization scale independence in Sect. 3. Section 4 is devoted to the general properties of the $\pi\eta$ scattering amplitude. We discuss the kinematics, the definition of suitable "good" observables, the dispersion representation of the amplitude and the construction of the bare expansion. Various possibilities of its reparametrization are described in a detailed way in Sect. 5. A numerical illustration of the particular variants is made in Sect. 6, where we also numerically illustrate the subtleties of the construction of the bare expansion. We recapitulate the results of the standard variant of χPT and compare them with the resummed approach. We concentrate on the dependence on the LECs as well as on the sensitivity to the higher order remainders and make an attempt to estimate their values using a matching with GχPT and a simple version of resonance chiral theory. Section 7 contains a summary and conclusions. Some technical details are postponed to the appendices.

[4] Effectively generalized chiral power counting means partial resummation of these terms.

2 Resummation of the vacuum fluctuations – motivation and basic notation

As we have mentioned in the Introduction, the potentially large vacuum fluctuations of the $\bar{s}s$ pairs might result in instabilities of the chiral expansion, which originate in the possibility that for some observables the next-to-leading order correction could be numerically comparable with the leading order one. As discussed in [8, 9], this could generally cause problems with the convergence of the formal chiral expansion. Nevertheless, at least for some carefully defined "good" observables, it is natural to presume some sort of satisfactory convergence properties. Such "good" observables are assumed to be those which can be obtained directly from the low energy correlation functions in the domain of their analyticity far away from singularities and which are *linearly* related to the corresponding correlator [8, 9]. Typical examples are the squares of the PGB decay constants F_P^2, the products $F_P^2 M_P^2$ where M_P are the PGB masses and also the subthreshold parameters which can be derived from the products $A\prod_{i=1}^4 F_{P_i}$, where A is the PGB scattering amplitude $1+2\to 3+4$. Let us write the expansion of such a "good" observable G in the form of a (carefully defined) *bare expansion* [8] as[5],

$$G = G^{(2)} + G^{(4)} + G\delta_G\,, \tag{17}$$

where $G^{(2)} = g^{(2)}(F_0, B_0, m_q)$ and $G^{(4)} = g^{(4)}(F_0, B_0, m_q, L_i, M_P^2)$ correspond to the sum of the leading and next-to-leading order terms respectively, and the renormalization scale independent quantity δ_G accommodates the higher order remainders.

A terminological note: in what follows we use the term *strict* chiral expansion for an unmodified expansion in terms of the LECs strictly respecting the chiral orders. The *bare* expansion, though still expressed in terms of LECs, accumulates some modifications dictated by physical requirements. It is the bare expansion which is assumed to be globally convergent.

For a "good" observable it is then assumed that

$$|\delta_G| \ll 1\,, \tag{18}$$

as a natural assumption. This property of the bare chiral expansion (17) is called global convergence in [8, 9]. Note, however, that the validity of the inequality (18) might depend on the definition of the remainder δ_G, which is not fixed unambiguously and might differ according to the calculation scheme in use. We will comment on this point later on.

The above mentioned possible instability in (17) appears when $G^{(2)} \sim G^{(4)}$, i.e. $X_G \lesssim 1$, where

$$X_G = \frac{G^{(2)}}{G}\,. \tag{19}$$

[5] Here we tacitly assume the standard chiral power counting. An analogous expansion could be written also for the generalized case.

Such an instability manifests itself in the expansion of the observables depending on G nonlinearly [8]. For instance, for a ratio of two "good" observables G and G' formally expanded in the form (17),

$$\frac{G}{G'} = \left(\frac{G^{(2)}}{G'^{(2)}}\right) + \left(\frac{G^{(2)}}{G'^{(2)}}\right)\left(\frac{G^{(4)}}{G^{(2)}} - \frac{G'^{(4)}}{G'^{(2)}}\right) + \frac{G}{G'}\delta_{G/G'}\,, \tag{20}$$

we get for the remainder $\delta_{G/G'}$

$$\delta_{G/G'} = \frac{(1-X_{G'})(X_G - X_{G'})}{X_G'^2} + \frac{\delta_G}{X_{G'}} - \frac{X_G\delta_{G'}}{X_G'^2}\,. \tag{21}$$

For $X_{G'} \lesssim 1$ this might be numerically large even if both $|\delta_G|$, $|\delta_{G'}|$ were reasonably small. In this sense, the ratio of two globally convergent observables need not to be necessarily globally convergent too. It should therefore be much safer not to expand such "dangerous" observables and rather write the ratio in the "resummed" form

$$\frac{G}{G'} = \frac{G^{(2)} + G^{(4)}}{G'^{(2)} + G'^{(4)}} + \frac{G}{G'}\tilde{\delta}_{G/G'}\,. \tag{22}$$

Equation (22) is an exact algebraic identity provided we explicitly keep the remainder:

$$\tilde{\delta}_{G/G'} = \frac{\delta_G - \delta_{G'}}{1-\delta_{G'}}\,. \tag{23}$$

In this case $\tilde{\delta}_{G/G'}$ remains for $|\delta_G|, |\delta_{G'}| \ll 1$ under numerical control.

Of course, only the fact that the bare expansion of some observable is not globally convergent does not necessarily correspond to the collapse of the convergence, because the next-to-next-to-leading order $G^{(6)}$ can saturate the series in such a way that the next-to-next-to-leading remainder

$$G\delta_G^{\mathrm{NNLO}} = G - G^{(2)} - G^{(4)} - G^{(6)} \tag{24}$$

is reasonably small. Namely, this is the usual assumption behind the $O(p^6)$ calculations. Violation of the global convergence property here means merely that the $O(p^6)$ contribution has an unnatural size, i.e. $G^{(6)} \lesssim G^{(2)} + G^{(4)}$. This could, however, destabilize the $O(p^6)$ chiral expansion of ratios in the way similar to that discussed above.

Provided we allow for the expansion of the "good" observables only, we are also pressed to modify the next step leading from the bare expansion to the usual output of χPT, consisting of a reparametrization of the expansion by expressing some of the LECs in terms of the physical observables such as masses and PGB decay constants. This step converts the series into an expansion in powers and logs of the (squared) PGB masses instead of quark masses. To achieve this, it is either necessary to invert a bare chiral expansion of some observable (in the case of the $O(p^2)$ LECs) or to use an observable which might be generally a "dangerous" one. Let us briefly discuss the first case. Schematically, suppose that some $O(p^2)$ LEC G_0 (e.g. F_0^2) just corresponds to the leading term $G^{(2)}$ of the expansion of the observable G. Then we can write the algebraic identity

$$G_0 = G - G^{(4)}(G_0) - G\delta_G, \tag{25}$$

where we explicitly point out the dependence of the next-to-leading term on G_0. To convert this expansion and express G_0 by means of the series in G, one substitutes G for G_0 on the right hand side. This defines a new remainder δ_{G_0}

$$G_0 = G - G^{(4)}(G) + G_0\delta_{G_0}\,, \tag{26}$$

for which we get

$$\delta_{G_0} = -\frac{1-X_G}{X_G} + \frac{1}{X_G}\frac{G^{(4)}(G)}{G}\,. \tag{27}$$

This could cause an instability of the converted expansion for G_0 in terms of G for $X_G \lesssim 1$ even if the relative size of the next-to-leading order $G^{(4)}(G)/G$ is reasonably small, irrespective of the condition for global convergence $|\delta_G| \ll 1$.

On the other hand, suppose that some $O(p^4)$ constant G_1 coincides with the next-to-leading term $G^{(4)}$. In this case we have the algebraic identity for G_1

$$G_1 = G - G^{(2)} - G\delta_G \tag{28}$$

and the remainder here is perfectly under control, provided G has a globally convergent bare expansion and we *do not* re-express $G^{(2)}$ in terms of physical observables (i.e. provided we treat the $O(p^2)$ LECs as free parameters).

From the above simple considerations it follows that in order to avoid potential problems with the instabilities of the chiral expansion, which might be present in the three-flavor χPT in the case of small $X(3)$ and $Z(3)$ (cf. (6) and (7)), we should [8, 9]:

- carefully define the bare expansion;
- confine ourselves (as far as the bare chiral expansion is concerned) to the linear space of "good" observables and keep the "dangerous" observables in the non-perturbative "resummed" form;
- use rather $\Sigma(3)$, $F_0(3)$ (or $X(3)$ and $Z(3)$) and $r = 2m_s/(m_u+m_d)$ as free parameters[6] instead of expressing them in the form of the series in PGB masses and decay constants;
- eliminate the $O(p^4)$ LECs algebraically, using bare expansions of "good" observables such as F_P^2, $F_P^2M_P^2$.[7]

In the next section we shall illustrate the possible subtleties of the first step of this general recipe on the concrete example[8] of the PGB scattering amplitude $P_1P_2 \to P_3P_4$.

[6] Note that r is related to the "dangerous" observable

$$2\frac{F_K^2M_K^2}{F_\pi^2M_\pi^2} - 1 = r + \ldots$$

[7] We will do this for L_4–L_8 but leave L_1–L_3 free, and also L_7 is a special case, as we see in the following.

[8] We shall tacitly assume the case of three light flavors in what follows.

3 Construction of the bare expansion for the scattering $P_1P_2 \to P_3P_4$

3.1 Chiral expansion of the "good" observables

Let us assume scattering of pseudoscalar mesons $P_1P_2 \to P_3P_4$ with masses M_{P_i}. The amplitude $S(s,t;u)$ is defined as

$$\begin{aligned}\langle P_3(k_3)P_4(k_4)_{\text{out}}|P_1(k_1)P_2(k_2)_{\text{in}}\rangle \\ = \mathrm{i}(2\pi)^4\delta^{(4)}(k_3+k_4-k_1-k_2)S(s,t;u),\end{aligned} \tag{29}$$

where s, t and u are the usual Mandelstam variables. The amplitude is related to the "good" observable[9]

$$G(s,t;u) = \prod_{i=1}^{4} F_{P_i} S(s,t;u) \tag{30}$$

(where F_{P_i} are the decay constants) which can directly be obtained from the (cut) four-point function of the axial currents. Let us write for $G(s,t;u)$ the following *strict* chiral expansion in terms of the low energy constants:

$$G = G^{(2)} + G^{(4)}_{\text{ct}} + G^{(4)}_{\text{tad}} + G^{(4)}_{\text{unit}} + G\delta_G\,. \tag{31}$$

$G\delta_G$ accommodates the higher order remainders. Using the functional method, G can be obtained from the generating functional

$$\begin{aligned}F_0^4 Z[U,v,p,a,s] = F_0^4 \int \mathrm{d}^4x(&\mathcal{L}^{(2)}(U,v,p,a,s) \\ &+ \mathcal{L}^{(4)}(U,v,p,a,s)) \\ &+ F_0^4 Z^{(4)}_{\text{loop}}[U,v,p,a,s] + \ldots\end{aligned} \tag{32}$$

by setting $v = s = p = 0$, $s = 2B_0\mathcal{M}$ and expanding in the fields Φ, where $U = \exp(\mathrm{i}\Phi/F_0)$. Following the notation in [3], we have

$$\begin{aligned}Z^{(4)}_{\text{loop}}[U,v,p,a,s] &= Z^{(4)}_{\text{tad}}[U,v,p,a,s] + Z^{(4)}_{\text{unit}}[U,v,p,a,s] \\ &= \frac{\mathrm{i}}{2}\ln\det D_0 + \frac{\mathrm{i}}{4}\mathrm{Tr}\left(D_0^{-1}\delta\right) \\ &\quad - \frac{\mathrm{i}}{4}\mathrm{Tr}\left(D_0^{-1}\delta D_0^{-1}\delta\right) + \ldots\end{aligned} \tag{33}$$

In the above formulae,

$$D_0^{ab} = \delta^{ab}\Box + \frac{1}{2}B_0\mathrm{tr}(\{\lambda^a,\lambda^b\}\mathcal{M})\,, \tag{34}$$

and $\mathcal{M}$ is the quark mass matrix. Note that this representation of $Z^{(4)}_{\text{loop}}$ assumes that the masses running in the loops are the $O(p^2)$ masses rather than the physical masses. Or, in more detail, provided we start with the chiral expansion of the squared product of the masses and decay constants

$$F_P^2M_P^2 = \left(F_P^2M_P^2\right)^{(2)} + \left(F_P^2M_P^2\right)^{(4)} + F_P^2M_P^2\delta_{FM_P}\,, \tag{35}$$

[9] Strictly speaking, the "good" observables correspond to the subthreshold parameters derived form $G(t,s;u)$ in an unphysical point away from singularities.

the masses in the loops are defined as

$$\overset{o}{M}{}_P^2 = \frac{\left(F_P^2 M_P^2\right)^{(2)}}{F_0^2} \,. \tag{36}$$

Note, however, that this is the first term in a potentially "dangerous" expansion of the ratio

$$M_P^2 = \frac{F_P^2 M_P^2}{F_P^2} = \frac{\left(F_P^2 M_P^2\right)^{(2)} + \left(F_P^2 M_P^2\right)^{(4)} + F_P^2 M_P^2 \delta_P}{F_0^2 + (F_P^2)^{(4)} + F_P^2 \delta_{F_P}}$$
$$= \overset{o}{M}{}_P^2 + \ldots \tag{37}$$

From this definition of $Z^{(4)}_{\text{loop}}$ we obtain $G^{(4)} = G^{(4)}_{\text{ct}} + G^{(4)}_{\text{tad}} + G^{(4)}_{\text{unit}}$, which is exactly renormalization scale independent even for the external momenta off-shell. This meets the requirement of the renormalization scale independence of the remainder δ_G.

The first two terms of the above strict chiral expansion for $G(s,t;u)$ have a serious drawback in the sense that the singularities in the complex stu planes required by unitarity are not placed at the physical thresholds but rather at points given by the leading order terms $\overset{o}{M}_P$ of the chiral expansion of the PGB masses. Straightforward substitution $\overset{o}{M}_P \to M_P$ in the propagators of the loops, which apparently means merely a redefinition of the remainder δ_G, could, however, in general spoil its exact renormalization scale independence. It is therefore desirable to use the freedom in the definition of the remainder more carefully in order to reconcile both scale independence of $G^{(4)}$ and unitarity. For this purpose, a useful tool is the matching with a dispersive representation [8] of the amplitude $S(s,t;u)$ based on the reconstruction theorem [25, 29].

3.2 Dispersive representation for $G(s,t;u)$

The above mentioned reconstruction theorem for the PGB scattering amplitude is based on the basic properties of unitarity, analyticity and crossing symmetry and provides us with the most general form of the PGB scattering amplitude up to the order $O(p^6)$ in terms of dispersive integrals with known discontinuities. It was first proved for the case of $\pi\pi$ scattering in [25, 29] and for πK scattering in [32, 39], and since then it has been intensively used in various contexts. Here we use the general form of the theorem, a more detailed discussion of which will be presented elsewhere [40].

For the scattering of pseudoscalar mesons $P_1P_2 \to P_3P_4$, let us denote the s-, t- and u-channel amplitudes by $S(s,t;u)$, $T(s,t;u)$ and $U(s,t;u)$ and write their partial wave expansion as

$$A(s,t;u) = 32\pi \sum_{l=0}^{\infty} (2l+1) A_l(s) P_l(\cos\theta_A) \,, \tag{38}$$

where $A = S, T, U$ and

$$\cos\theta_A = \frac{s(t-u) + \Delta_{A_\text{i}} \Delta_{A_\text{f}}}{\lambda^{1/2}_{A_\text{i}}(s) \lambda^{1/2}_{A_\text{f}}(s)} \,. \tag{39}$$

Here $A_l(s)$ are the partial waves,

$$\lambda_{A_{\text{i,f}}}(s) = \left(s - (M_{P_j} + M_{P_k})^2\right)\left(s - (M_{P_j} - M_{P_k})^2\right) \tag{40}$$

is the triangle function which corresponds to the initial/final state $A_{\text{i,f}}$ (consisting of the pseudoscalars P_jP_k) of the process in the channel A and

$$\Delta_{A_{\text{i,f}}} = M^2_{P_j} - M^2_{P_k} \,. \tag{41}$$

According to the theorem, we get the following representation for the amplitude $S(s,t;u)$:

$$S(s,t;u) = \mathcal{S}(s,t;u) + \mathcal{S}_{\text{unit}}(s,t;u) + O(p^8) \,, \tag{42}$$

where $\mathcal{S}(s,t;u)$ is a third order polynomial with the same symmetries as the whole amplitude $S(s,t;u)$. The nontrivial analytical properties are incorporated in the unitarity part $\mathcal{S}_{\text{unit}}(s,t;u)$, which can be expressed as

$$\begin{aligned}\mathcal{S}_{\text{unit}}(s,t;u) &= \Phi^S(s) + \Phi^T(t) + \Phi^U(u) \\ &+ [s(t-u) + \Delta_{12}\Delta_{34}]\Psi^S(s) \\ &+ [t(s-u) + \Delta_{13}\Delta_{24}]\Psi^T(t) \\ &+ [u(t-s) + \Delta_{14}\Delta_{23}]\Psi^U(u) \,.\end{aligned} \tag{43}$$

In the last expression, $\Delta_{ij} = M^2_{P_i} - M^2_{P_j}$. The functions $\Phi^A(s)$ and $\Psi^A(s)$ with $A = S, T, U$ are analytic in the cut complex plane with the right hand cut from $\tau_A = \min_{i,j}(M_{P_i} + M_{P_j})^2$ (where P_iP_j are the possible intermediate states in the given channel A) to infinity with discontinuities given by

$$\text{disc}\,\Phi^A(s) = 32\pi\theta(s-\tau_A)\text{disc}\,A_0(s) \,, \tag{44}$$
$$\text{disc}\,\Psi^A(s) = 96\pi\theta(s-\tau_A)\text{disc}\frac{A_1(s)}{\lambda^{1/2}_{A_\text{i}}(s)\lambda^{1/2}_{A_\text{f}}(s)} \,. \tag{45}$$

Here $A_0(s)$ and $A_1(s)$ are the corresponding $l = 0, 1$ partial waves.

Consequently, once the right hand sides of (44) and (45) are known, the unitarity part $\mathcal{S}_{\text{unit}}(s,t;u)$ of the amplitude can be uniquely reconstructed to $O(p^6)$ up to the polynomial, which encompasses subtraction polynomials for the dispersion integrals.

Let us now assume that the chiral expansion of the amplitudes can be cast in the form:

$$A(s,t;u) = A^{(2)}(s,t;u) + A^{(4)}(s,t;u) + A\delta_A \,, \tag{46}$$
$$A^{(n)}(s,t;u) = 32\pi \sum_{l=0}^{\infty} (2l+1) A^{(n)}_l(s) P_l(\cos\theta_s(t)) \,. \tag{47}$$

Starting from the $O(p^2)$ amplitudes, we can use the two particle partial wave unitarity to get the discontinuity of the partial waves $A_l^{(4)}(s)$ along the right hand cut:[10]

$$\mathrm{disc}\, A_l^{(4)}(s) = \sum_{ij} \frac{2}{z_{ij}} \frac{\lambda_{ij}^{1/2}(s)}{s} A_l^{(2)ij\to A_\mathrm{f}}(s) A_l^{(2)ij\to A_\mathrm{i}}(s)^* + O(p^6)\,. \tag{48}$$

Here $z_{ij} = 1, 2$ is a symmetry factor taking into account the possibility of identical particles in the intermediate state ij. Inserting this into the dispersive integrals we easily[11] get a minimal form for the $O(p^4)$ unitarity corrections in terms of the functions $\Phi^{(4)A}(s)$ and $\Psi^{(4)A}(s)$ reconstructed from the $O(p^2)$ amplitudes

$$\Phi^{(4)A}(s) = (32\pi)^2 \sum_{ij} \frac{1}{z_{ij}} \overline{\overline{J}}_{ij}(s) A_0^{(2)A_\mathrm{i}\to ij}(s) A_0^{(2)ij\to A_\mathrm{f}}(s)^*, \tag{49}$$

$$\Psi^{(4)A}(s) = \frac{(96\pi)^2}{3} \sum_{ij} \frac{1}{z_{ij}} \overline{\overline{J}}_{ij}(s) \times \frac{A_1^{(2)A_\mathrm{i}\to ij}(s) A_1^{(2)ij\to A_\mathrm{f}}(s)^*}{\lambda_{A_\mathrm{i}}^{1/2}(s)\lambda_{A_\mathrm{f}}^{1/2}(s)}\,. \tag{50}$$

$\overline{\overline{J}}_{ij}(s) = J_{ij}^r(s) - J_{ij}^r(0) - sJ_{ij}^{r'}(s)$ corresponds to the twice subtracted scalar bubble with internal line masses M_{P_i,P_j}. Provided $\Delta_{A_\mathrm{i}} = 0$ or $\Delta_{A_\mathrm{f}} = 0$, which will be our case, it can be shown that we only need one subtraction, $\overline{J}_{ij}(s) = J_{ij}^r(s) - J_{ij}^r(0)$ instead of $\overline{\overline{J}}_{ij}(s)$. The explicit form of the function $\overline{J}_{ij}(s)$ is given in the Appendix D.

The above formulae can be used to write a dispersive representation of the "good" observable $G(s,t;u) = \prod_{i=1}^4 F_{P_i} S(s,t;u)$ to the next-to-leading order in the form

$$G(s,t;u) = \mathcal{G}(s,t;u) + \mathcal{G}_\mathrm{unit}(s,t;u)\,, \tag{51}$$

where $\mathcal{G}(s,t;u)$ is the polynomial part, and the unitarity corrections up to $O(p^6)$ are included in

$$\begin{aligned}\mathcal{G}_\mathrm{unit}(s,t;u) = {} & \phi^S(s) + \phi^T(t) + \phi^U(u) \\ & + [s(t-u) + \Delta_{12}\Delta_{23}]\psi^S(s) \\ & + [t(s-u) + \Delta_{13}\Delta_{24}]\psi^T(t) \\ & + [u(t-s) + \Delta_{14}\Delta_{23}]\psi^U(u)\,.\end{aligned} \tag{52}$$

Our goal is to write down a representation of $\phi^{(4)A}$ and $\psi^{(4)A}$, which, one may notice, are quantities distinct from $\Phi^{(4)A}$ and $\Psi^{(4)A}$, analogous to (49) and (50). Note, however, that while the relation of $G(s,t;u)$ and $S(s,t;u)$ is unambiguously fixed to all orders by (30), the amplitude can be defined order by order in various ways. For example, for the "good" observable G, the leading order piece $G^{(2)}$ of its strict chiral expansion is fixed by the lowest order Lagrangian $\mathcal{L}^{(2)}$, but the corresponding $O(p^2)$ piece of the amplitude S can be related in various ways. Similarly, the same is true order by order, where the amplitude at the given order can be defined up to higher order corrections.

The most straightforward way is to write a safe expansion for $S(s,t;u)$ in the form

$$S(s,t;u) = \left(\prod_{i=1}^4 F_{P_i}\right)^{-1} \times (G^{(2)}(s,t;u) + G^{(4)}(s,t;u) + G\delta_G)\,, \tag{53}$$

with physical values of F_{P_i}, thus satisfying the relation (30) order by order:

$$S^{(n)}(s,t;u) = \left(\prod_{i=1}^4 F_{P_i}\right)^{-1} G^{(n)}(s,t;u)\,. \tag{54}$$

As we shall see, the minimal modification of the form derived from the generating functional is obtained by using an alternative, potentially "dangerous" expansion:

$$\begin{aligned}S(s,t;u) &= \left(\prod_{i=1}^4 F_{P_i}\right)^{-1} G(s,t;u) \\ &= \left(\prod_{i=1}^4 F_0\left(1 + \frac{1}{2}\frac{\left(F_{P_i}^2\right)^{(4)}}{F_0^2} + \ldots\right)\right)^{-1} \\ &\quad \times (G^{(2)}(s,t;u) + G^{(4)}(s,t;u) + \ldots) \\ &= F_0^{-4} G^{(2)}(s,t;u) - \frac{1}{2} F_0^{-6} G^{(2)}(s,t;u) \\ &\quad \times \sum_{i=1}^4 \left(F_{P_i}^2\right)^{(4)} + F_0^{-4} G^{(4)}(s,t;u) + \ldots\,,\end{aligned} \tag{55}$$

which defines

$$\tilde{S}^{(2)}(s,t;u) = F_0^{-4} G^{(2)}(s,t;u)\,, \tag{56}$$

$$\tilde{S}^{(4)}(s,t;u) = F_0^{-4} G^{(4)}(s,t;u) - \frac{1}{2} F_0^{-6} G^{(2)}(s,t;u) \sum_{i=1}^4 (F_{P_i}^2)^{(4)}\,. \tag{57}$$

The representation of $\phi^{(4)A}$ and $\psi^{(4)A}$ is therefore not unique. According to our definitions of the amplitude we get either (we assume a partial wave expansion of $G(s,t;u)$ analogous to (38))

$$\phi^{(4)A}(s) = (32\pi)^2 \sum_{ij} \frac{1}{z_{ij}} \frac{\overline{\overline{J}}_{ij}(s)}{F_{P_i}^2 F_{P_j}^2} \times G_0^{(2)A_\mathrm{i}\to ij}(s) G_0^{(2)ij\to A_\mathrm{f}}(s)^*\,, \tag{58}$$

[10] It can be shown that more than two particle intermediate states yield a contribution of the order $O(p^8)$ and higher.

[11] Note that $A^{(2)}(s,t;u)$ are real polynomials of the first order in s, t and u.

$$\psi^{(4)A}(s) = \frac{(96\pi)^2}{3}\sum_{ij}\frac{1}{z_{ij}}\frac{\overline{\overline{J}}_{ij}(s)}{F_{P_i}^2 F_{P_j}^2} \times \frac{G_1^{(2)A_\mathrm{i}\to ij}(s)G_1^{(2)ij\to A_\mathrm{f}}(s)^*}{\lambda_{A_\mathrm{i}}^{1/2}(s)\lambda_{A_\mathrm{f}}^{1/2}(s)}\,, \qquad (59)$$

corresponding to the definition (54) or

$$\tilde{\phi}^{(4)A}(s) = (32\pi)^2 F_0^{-4}\sum_{ij}\frac{1}{z_{ij}}\overline{\overline{J}}_{ij}(s) \times G_0^{(2)A_\mathrm{i}\to ij}(s)G_0^{(2)ij\to A_\mathrm{f}}(s)^*\,, \qquad (60)$$

$$\tilde{\psi}^{(4)A}(s) = \frac{(96\pi)^2}{3}F_0^{-4}\sum_{ij}\frac{1}{z_{ij}}\overline{\overline{J}}_{ij}(s) \times \frac{G_1^{(2)A_\mathrm{i}\to ij}(s)G_1^{(2)ij\to A_\mathrm{f}}(s)^*}{\lambda_{A_\mathrm{i}}^{1/2}(s)\lambda_{A_\mathrm{f}}^{1/2}(s)}\,, \qquad (61)$$

when reconstructing the bare expansion of G from the "dangerous" expansion (55) and using the definitions (56) and (57) for the $O(p^2)$ and $O(p^4)$ amplitudes.

3.3 Matching the strict chiral expansion to the dispersive representation

The dispersive representation (51) can now be matched to (31). As we have mentioned above, the positions of the cuts in (31) and (51) are not the same; in the former case they correspond to the $O(p^2)$ masses (36), which ensures renormalization scale independence, while in the latter case they are determined by the physical ones, as required by the unitarity conditions. In order to reconcile both these requirements, one may proceed as follows (c.f. also [8]).

In (31), the nonanalytic terms are generally of the form $P(s)J_{ij}^r(s)$, where $J_{ij}^r(s)$ is the renormalized scalar bubble defined in Appendix D, and $P(s)$ is some second order polynomial. As a first step, one rewrites these expressions in terms of $\overline{\overline{J}}_{ij}(s)$, writing $J_{ij}^r(s) = J_{ij}^r(0) + s\overline{J}'_{ij}(0) + \overline{\overline{J}}_{ij}(s)$. This adjustment allows us to split G uniquely into a polynomial part G_pol and a nonanalytic part G_cut, which incorporates the unitarity cuts

$$G(s,t;u) = G_\mathrm{pol}(s,t;u) + G_\mathrm{cut}(s,t;u) + G\delta_G\,, \quad (62)$$

where

$$G_\mathrm{pol}(s,t;u) = (G(s,t;u) - G\delta_G)|_{\overline{J}_{ij}=\overline{\overline{J}}_{ij}=0}\,. \quad (63)$$

Both parts are now renormalization scale independent.

As a second step, we replace the $G_\mathrm{cut}(s,t;u)$ with $\mathcal{G}_\mathrm{unit}(s,t;u)$ from (51). This means that we write

$$G(s,t;u) = G_\mathrm{pol}(s,t;u) + \mathcal{G}_\mathrm{unit}(s,t;u) + G\delta'_G\,, \quad (64)$$

where δ'_G is a new remainder defined by this equation. According to naive chiral power counting, $G_\mathrm{cut}(s,t;u) - \mathcal{G}_\mathrm{unit}(s,t;u) = O(p^6)$.

The third step, not necessary from the point of view of preserving unitarity and renormalization scale invariance, consists of a further modification of $\overset{o}{G}_\mathrm{pol}(s,t;u)$ by means of replacement of the $O(p^2)$ masses $\overset{o}{M}{}_P^2$ in $J_{ij}^r(0)$ with the physical masses M_P^2. This replacement does not spoil the renormalization scale independence of the $G_\mathrm{pol}(s,t;u)$ and corresponds to the convention introduced in [8,9]. This again means a redefinition of the remainders δ'_G, i.e. reshuffling of the terms of the next-to-next-to-leading order.

Note that the origin of the $J_{ij}^r(0)$ in one loop generating functional (33) is twofold: they may stem either from the tadpole part $Z_\mathrm{tad}^{(4)}$ or from the unitarity corrections $Z_\mathrm{unit}^{(4)}$. It was argued in [9] that in the former case the above mentioned replacement does not necessarily modify the numerical value of the remainders much. The reason should be that the chiral logs appear only in the combination $\mu_P \propto \overset{o}{M}{}_P^2 \ln(\overset{o}{M}{}_P^2/\mu^2)$. The replacement here means

$$\overset{o}{M}{}_P^2 \ln\left(\overset{o}{M}{}_P^2/\mu^2\right) \to \overset{o}{M}{}_P^2 \ln\left(M_P^2/\mu^2\right)\,. \quad (65)$$

Because $\overset{o}{M}{}_P^2 \propto Y = X/Z$, the difference should therefore either be small for $Y\sim 1$ (where $\overset{o}{M}{}_P^2 \sim M_P^2$) or the contribution of μ_P itself is tiny for $Y\to 0$.

On the other hand, the logs from $Z_\mathrm{unit}^{(4)}$ do not generally come with such a prefactor. Therefore, with the replacement $\overset{o}{M}{}_P^2 \to M_P^2$ inside $J_{ij}^r(0)$, one might create large differences between the "old" and "new" remainders due to the enhancement of the contributions of chiral logs for small Y. However, without the replacement inside the chiral logs of this type we could expect an unphysical increase (and irregularities) of the observables for $Y\to 0$. Also, here the replacement is natural physically; remember that the matching with the dispersive representation consists essentially of an analogous replacement within the unitary corrections. Let us also note that the splitting of the generating functional into the tadpole and unitarity part is not unique (it depends e.g. on the parametrization of the fluctuations around the classical solution of the $O(p^2)$ field equations in the functional integral), though the sum must be independent on this and therefore it is more consistent to use the same rule for both.[12] Nevertheless, it could be worthwhile to test the differences between various treatments of the chiral logs numerically (see Sect. 6.2).

The resulting *bare* expansion (64) now not only meets the requirement of the exact scale independence of the remainder δ'_G, it also has the correct physical location of the unitarity cuts. Of course, we could achieve the latter property simply by inserting physical masses into the functions

[12] Also notice that the offending Y dependence of the chiral logs with $O(p^2)$ masses inside always comes in the combination Y/μ^2 where μ is the renormalization scale. Provided we were able to reparametrize the bare expansion in such a way that all the running $O(p^4)$ constants were completely expressed in terms of the physical observables, the explicit independence on μ would at the same time guarantee elimination of the irregularities for $Y\to 0$. Such a treatment has to include the reparametrization of L_1–L_3, which is, however, beyond the scope of our paper.

$\overline{\overline{J}}_{ij}(s)$ in $G_{\mathrm{cut}}(s,t;u)$. The replacement $G_{\mathrm{cut}}(s,t;u) \to \mathcal{G}_{\mathrm{unit}}(s,t;u)$ has, however, another advantage. Namely, using the prescription (58) and (59), the corresponding amplitude, written in the form (without any expansion of the denominator)

$$S(s,t;u) = \frac{G(s,t;u)}{\prod_{i=1}^{4} F_{P_i}} \tag{66}$$

satisfies the relations of perturbative unitarity (with $S^{(2)}$ and $S^{(4)}$ given by (54))

$$\operatorname{disc} S_l^{(4)}(s) = \sum_{ij} \frac{2}{z_{ij}} \frac{\lambda_{ij}^{1/2}(s)}{s} S_l^{(2)ij\to A_{\mathrm{f}}}(s) S_l^{(2)ij\to A_{\mathrm{i}}}(s)^* , \tag{67}$$

exactly (i.e. not only modulo the next-to-next-to-leading correction), which may sometimes be technically useful (e.g. for the unitarization by means of the inverse amplitude method [41]). The same is true using the prescription (60) and (61) with $\tilde{S}^{(2)}$ and $\tilde{S}^{(4)}$ given by (56) and (57). As we shall see below, the latter prescription gives a minimal modification of the strict expansion (31) compatible with exact perturbative unitarity.

4 General properties of $\pi\eta$ scattering amplitude

4.1 Basic notation

Let us denote the s- and u-channel amplitude in the isospin conservation limit by

$$\langle \pi^b(p^b)\eta(q)_{\mathrm{out}} | \pi^a(p^a)\eta(p)_{\mathrm{in}} \rangle = \mathrm{i}(2\pi)^4 \delta(P_f - P_i) \times \delta^{ab} S(s,t;u) \tag{68}$$

and the crossed amplitude in the t-channel by

$$\langle \eta(p)\eta(q)_{\mathrm{out}} | \pi^a(p^a)\pi^b(p^b)_{\mathrm{in}} \rangle = \mathrm{i}(2\pi)^4 \delta(P_f - P_i) \times \delta^{ab} T(s,t;u) . \tag{69}$$

Crossing and Bose symmetries then yield

$$\begin{aligned} T(s,t;u) &= S(t,s;u) , \\ S(s,t;u) &= S(u,t;s) , \\ T(s,t;u) &= T(s,u;t) . \end{aligned} \tag{70}$$

Writing the partial wave expansion as

$$\begin{aligned} S(s,t;u) &= 32\pi \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta_s) S_l(s) , \\ \cos\theta_s &= \frac{(t-u)s + \Delta_{\eta\pi}^2}{\lambda_{\eta\pi}(s)} , \end{aligned} \tag{71}$$

the scattering lengths a_l and phase shifts $\delta_l(s)$ are given by

$$\begin{aligned} \operatorname{Re} S_l(s) &= \frac{\sqrt{s}}{4} P^{2l} (a_l + O(P^2)) \\ &\quad \text{for} \quad P \to 0 , \quad s \to (M_\eta + M_\pi)^2 \\ \delta_l(s) &= \arctan\left(\frac{4P}{\sqrt{s}} \operatorname{Re} S_l(s) \right) , \end{aligned} \tag{72}$$

where $P = \lambda_{\eta\pi}^{1/2}(s)/2\sqrt{s}$ is the CMS momentum. That is, in units of (pion Compton wavelength)$^{2l+1}$, we have

$$a_l = M_\pi^{2l+1} \lim_{P\to 0} \frac{4}{\sqrt{s}P^{2l}} \operatorname{Re} A_l(s) . \tag{73}$$

Let us also define the subthreshold parameters c_{ij} in terms of the expansion of the amplitude in the point of analyticity $t = 0$, $s = u = \Sigma_{\eta\pi} = M_\eta^2 + M_\pi^2$:

$$S(s,t;u) = \sum_{i,j} c_{ij} t^i \nu^{2j} , \tag{74}$$

where

$$\nu = \frac{s-u}{4M_\eta} = \frac{2s + t - 2\Sigma_{\eta\pi}}{4M_\eta} . \tag{75}$$

The dimension c_{ij} is $\dim[c_{ij}] = \text{mass}^{-2i-2j}$, we shall refer to the dimensionless numbers $c_{ij} M_\pi^{2i+2j}$ below. Let us note that in the limit $m_u = m_d = 0$ we have two Adler zeros, at $p^a = 0$ and $p^b = 0$, which implies the following $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}}$ theorem:

$$\lim_{m_u = m_d \to 0} c_{00} = 0 . \tag{76}$$

We can also quote the low energy current algebra result [42]

$$S(s,t;u) = \frac{M_\pi^2}{3F_\eta^2} , \tag{77}$$

which is in agreement with (76).

4.2 Dispersive representation

As a result of the symmetry properties of the amplitudes, the dispersive representation to the next-to-leading order (52) for $G_{\pi\eta}(s,t;u) = F_\pi^2 F_\eta^2 S(s,t;u)$ simplifies, namely $\phi^S = \phi^U \equiv \phi$ and $\psi^S = \psi^U \equiv \psi$. The intermediate states in (58) and (59) are[13] $\pi\eta$ and $\overline{K}K$ in the s- and u-channels and $\pi\pi$, $\eta\eta$ and $\overline{K}K$ in the t-channel. This implies $\psi(s) = O(p^6)$, because the P-waves in the s-channel start at $O(p^4)$ due to the low energy theorem (77) for the $\pi\eta \to \pi\eta$ amplitude and as a result of charge conjugation invariance of the $\pi\eta \to \overline{K}K$ amplitude. Moreover, $\psi^T = 0$, because the partial wave decomposition of the t-channel amplitude $T(s,t;u)$ contains only even partial waves due to Bose symmetry and charge conjugation. We therefore get

$$G_{\pi\eta}(s,t;u) = G_{\pi\eta,\mathrm{pol}}(s,t;u) + \mathcal{G}_{\pi\eta,\mathrm{unit}}(s,t;u) + O(p^6) , \tag{78}$$

[13] Here we assume isospin conservation.

where the polynomial part has the following general form:

$$G_{\pi\eta,\mathrm{pol}}(s,t;u)=\alpha+\beta t+\gamma t^2+\omega(s-u)^2\,. \qquad (79)$$

Note that the parameters $\alpha,\dots,\omega$ are related to the expansion of the Green function $G_{\pi\eta}(s,t;u)$ at the point of analyticity $t=0$, $s=u=\Sigma_{\eta\pi}=M_\eta^2+M_\pi^2$ and therefore they represent "good observables" according to our classification.

The dispersive part is

$$\mathcal{G}_{\pi\eta,\mathrm{unit}}(s,t;u)=\phi^T(t)+\phi(s)+\phi(u)\,, \qquad (80)$$

where $\phi^T(t)$ and $\phi(s)$ are given by (58). A complete list of relevant leading order contributions $G_{0,1}^{(2)12\to ij}$ and $G_{0,1}^{(2)ij\to 34}$ can be found in Appendix C; here we give the resulting expressions (transcription to the convention (60) and (61) is straightforward):

$$\begin{aligned}\phi(s)=F_0^4\Bigg\{&\frac{1}{9}\,\overset{o}{M}{}_\pi^4\,\frac{\overline{J}_{\pi\eta}(s)}{F_\pi^2F_\eta^2}\\&+\frac{3}{8}\Bigg[\left(s-\frac{1}{3}M_\eta^2-\frac{1}{3}M_\pi^2-\frac{2}{3}M_K^2\right)\\&-\frac{1}{3}\left(2\,\overset{o}{M}{}_K^2-\overset{o}{M}{}_\pi^2-\overset{o}{M}{}_\eta^2\right)\Bigg]^2\frac{\overline{J}_{KK}(s)}{F_K^4}\Bigg\}\,,\\ \phi^T(s)=F_0^4\Bigg\{&\frac{1}{3}\,\overset{o}{M}{}_\pi^2\left[\left(s-\frac{4}{3}M_\pi^2\right)+\frac{5}{6}\,\overset{o}{M}{}_\pi^2\right]\frac{\overline{J}_{\pi\pi}(s)}{F_\pi^4}\\&-\frac{1}{18}\,\overset{o}{M}{}_\pi^2\left(\overset{o}{M}{}_\pi^2-4\,\overset{o}{M}{}_\eta^2\right)\frac{\overline{J}_{\eta\eta}(s)}{F_\eta^4}\\&+\frac{1}{8}\left[\left(s-\frac{2}{3}M_\pi^2-\frac{2}{3}M_K^2\right)+\frac{2}{3}\left(\overset{o}{M}{}_K^2+\overset{o}{M}{}_\pi^2\right)\right]\\&\times\left[\left(3s-2M_K^2-2M_\eta^2\right)+\left(2\,\overset{o}{M}{}_\eta^2-\frac{2}{3}\,\overset{o}{M}{}_K^2\right)\right]\\&\times\frac{\overline{J}_{KK}(s)}{F_K^4}\Bigg\}\,. \end{aligned} \qquad (81)$$

In terms of these functions, we have (notice that $\phi^T(0)=0$)

$$\begin{aligned}a_0=&\frac{1}{8\pi F_\eta^2F_\pi^2}\frac{M_\pi}{(M_\pi+M_\eta)}\left(\alpha+16\omega M_\eta^2M_\pi^2\right.\\&\left.+\phi\left((M_\pi+M_\eta)^2\right)+\phi\left((M_\eta-M_\pi)^2\right)\right)\\ a_1=&\frac{1}{12\pi F_\eta^2F_\pi^2}\frac{M_\pi^3}{(M_\pi+M_\eta)}\Big(\beta+8\omega M_\eta M_\pi\\&+\phi^{T\prime}(0)-\phi'\left((M_\eta-M_\pi)^2\right)\Big)\end{aligned} \qquad (82)$$

and

$$\begin{aligned}c_{00}&=\frac{1}{F_\eta^2F_\pi^2}\left(\alpha+2\phi(\Sigma_{\eta\pi})\right)\,,\\ c_{10}&=\frac{1}{F_\eta^2F_\pi^2}\left(\beta+\phi^{\mathrm{T}\prime}(0)-\phi'(\Sigma_{\eta\pi})\right)\,,\\ c_{20}&=\frac{1}{F_\eta^2F_\pi^2}\left(\gamma+\frac{1}{2}\phi^{\mathrm{T}\prime\prime}(0)+\frac{1}{4}\phi''(\Sigma_{\eta\pi})\right)\,,\\ c_{01}&=\frac{16M_\eta^2}{F_\eta^2F_\pi^2}\left(\omega+\frac{1}{4}\phi''(\Sigma_{\eta\pi})\right)\,.\end{aligned} \qquad (83)$$

While the scattering lengths, being related to the value of the amplitude at the threshold, are not candidates for "good observables", the situation is a little bit more subtle in the case of the subthreshold parameters. Provided the η decay constant were known from experiments as accurately as F_π, then (similarly to $\alpha,\beta,\dots$) also the c_{ij} could be treated as "good observables". However, this is not the case, and we should rather use a chiral expansion of F_η in the above formulae. Therefore, the subthreshold parameters are typical examples of the dangerous ratios, which should be treated with care.

4.3 Bare expansion for $G(s,t;u)$

For a strict expansion in terms of LECs (i.e. without any reparametrization in terms of physical observables) derived from (32) and (33), we have confirmed the results of [38] by independent calculation. The $O(p^4)$ expansion can be written in the form

$$G_{\pi\eta}=G^{(2)}+G_{\mathrm{ct}}^{(4)}+G_{\mathrm{tad}}^{(4)}+G_{\mathrm{unit}}^{(4)}+G\delta_G\,, \qquad (84)$$

where

$$\begin{aligned}G^{(2)}(s,t;u)=&\frac{F_0^2}{3}\,\overset{o}{M}{}_\pi^2\,,\\ G_{\mathrm{ct}}^{(4)}(s,t;u)=&8\left(L_1^r(\mu)+\frac{1}{6}L_3^r(\mu)\right)\left(t-2M_\pi^2\right)\left(t-2M_\eta^2\right)\\&+4\left(L_2^r(\mu)+\frac{1}{3}L_3^r(\mu)\right)\\&\times\left[\left(s-M_\pi^2-M_\eta^2\right)^2+\left(u-M_\pi^2-M_\eta^2\right)^2\right]\\&+8L_4^r(\mu)\left[\left(t-2M_\pi^2\right)\overset{o}{M}{}_\eta^2+\left(t-2M_\eta^2\right)\overset{o}{M}{}_\pi^2\right]\\&-\frac{8}{3}L_5^r(\mu)\left(M_\pi^2+M_\eta^2\right)\overset{o}{M}{}_\pi^2+8L_6^r(\mu)\,\overset{o}{M}{}_\pi^2\\&\times\left(\overset{o}{M}{}_\pi^2+5\,\overset{o}{M}{}_\eta^2\right)+32L_7^r(\mu)\left(\overset{o}{M}{}_\pi^2-\overset{o}{M}{}_\eta^2\right)\\&\times\overset{o}{M}{}_\pi^2+\frac{64}{3}L_8^r(\mu)\,\overset{o}{M}{}_\pi^4\,,\\ G_{\mathrm{tad}}^{(4)}(s,t;u)=&-\frac{F_0^2}{3}\,\overset{o}{M}{}_\pi^2\left(3\mu_\pi+2\mu_K+\frac{1}{3}\mu_\eta\right)\,,\\ G_{\mathrm{unit}}^{(4)}(s,t;u)=&\frac{1}{9}\,\overset{o}{M}{}_\pi^4\left[J_{\pi\eta}^r(s)+J_{\pi\eta}^r(u)\right]\\&+\frac{3}{8}\left[s-M_\pi^2-M_\eta^2+\frac{2}{3}\,\overset{o}{M}{}_\pi^2\right]^2J_{KK}^r(s)\\&+\frac{3}{8}\left[u-M_\pi^2-M_\eta^2+\frac{2}{3}\,\overset{o}{M}{}_\pi^2\right]^2J_{KK}^r(u)\\&+\frac{1}{3}\,\overset{o}{M}{}_\pi^2\left[t-2M_\pi^2+\frac{3}{2}\,\overset{o}{M}{}_\pi^2\right]J_{\pi\pi}^r(t)\\&+\frac{2}{9}\,\overset{o}{M}{}_\pi^2\left(\overset{o}{M}{}_\eta^2-\frac{1}{4}\,\overset{o}{M}{}_\pi^2\right)J_{\eta\eta}^r(t)\\&+\frac{1}{8}\left[t-2M_\pi^2+2\,\overset{o}{M}{}_\pi^2\right]\\&\times\left[3t-6M_\eta^2+4\,\overset{o}{M}{}_\eta^2-\frac{2}{3}\,\overset{o}{M}{}_\pi^2\right]J_{KK}^r(t)\end{aligned} \qquad (85)$$

are the $O(p^2)$, counterterm, tadpole and unitarity contributions, respectively. In the above formulae, the masses within the loop functions $J^r_{PQ}(t)$ are the $O(p^2)$ masses

$$\overset{o}{M}{}^2_\pi = 2B_0\widehat{m}\,, \quad \overset{o}{M}{}^2_K = B_0\widehat{m}(r+1)\,, \quad \overset{o}{M}{}^2_\eta = \frac{2}{3}B_0\widehat{m}(2r+1)\,. \tag{86}$$

The chiral logs μ_P can be expressed using $J^r_{PP}(0)$:

$$\mu_P = \frac{\overset{o}{M}{}^2_P}{32\pi^2 F_0^2}\ln\frac{\overset{o}{M}{}^2_P}{\mu^2} = -\frac{\overset{o}{M}{}^2_P}{2F_0^2}\left(J^r_{PP}(0)+\frac{1}{16\pi^2}\right)\,. \tag{87}$$

Written in such a form, the sum $G^{(4)}_{\mathrm{ct}}+G^{(4)}_{\mathrm{tad}}+G^{(4)}_{\mathrm{unit}}$ is *exactly* renormalization scheme independent by construction. Let us now proceed as described in the previous section and write the bare expansion of $G(s,t;u)$ in the form

$$G_{\pi\eta}(s,t;u) = G_{\pi\eta,\mathrm{pol}}(s,t;u)+\mathcal{G}_{\pi\eta,\mathrm{unit}}(s,t;u)+G\delta'_G\,. \tag{88}$$

Writing $J^r_{ij}(s) = J^r_{ij}(0)+\overline{J}_{ij}(s)$ in (85), we get the renormalization scale independent polynomial part:

$$\begin{aligned} G_{\pi\eta,\mathrm{pol}}(s,t;u) = {} & G^{(2)}(s,t;u)+G^{(4)}_{\mathrm{ct}}(s,t;u)+G^{(4)}_{\mathrm{tad}}(s,t;u)\\ & +\frac{1}{3}\,\overset{o}{M}{}^2_\pi\left[t-2M^2_\pi+\frac{3}{2}\,\overset{o}{M}{}^2_\pi\right]J^r_{\pi\pi}(0)\\ & +\frac{2}{9}\,\overset{o}{M}{}^4_\pi\,J^r_{\pi\eta}(0)+\frac{2}{9}\,\overset{o}{M}{}^2_\pi\left(\overset{o}{M}{}^2_\eta-\frac{1}{4}\,\overset{o}{M}{}^2_\pi\right)\\ & \times J^r_{\eta\eta}(0)+\frac{3}{8}\Bigg\{\left[s-M^2_\pi-M^2_\eta+\frac{2}{3}\,\overset{o}{M}{}^2_\pi\right]^2\\ & +\left[u-M^2_\pi-M^2_\eta+\frac{2}{3}\,\overset{o}{M}{}^2_\pi\right]^2\\ & +\left[t-2M^2_\pi+2\,\overset{o}{M}{}^2_\pi\right]\\ & \times\left[t-2M^2_\eta+\frac{4}{3}\,\overset{o}{M}{}^2_\eta-\frac{2}{9}\,\overset{o}{M}{}^2_\pi\right]\Bigg\}J^r_{KK}(0)\,. \end{aligned} \tag{89}$$

Comparing this with the general form (79) of $G_{\pi\eta,\mathrm{pol}}(s,t;u)$, we get for the bare expansions of the parameters $\alpha-\omega$ the following manifestly renormalization scale independent form:

$$\begin{aligned} \alpha = {} & \frac{1}{3}F_0^2\,\overset{o}{M}{}^2_\pi+\frac{1}{96\pi^2}\,\overset{o}{M}{}^2_\pi\left(\frac{7}{2}\,\overset{o}{M}{}^2_\pi+\frac{11}{6}\,\overset{o}{M}{}^2_\eta\right)\\ & +4\left[8\left(L^r_1(\mu)+\frac{1}{6}L^r_3(\mu)\right)+\frac{3}{8}J^r_{KK}(0)\right]M^2_\pi M^2_\eta\\ & -\left[16L^r_4(\mu)+J^r_{KK}(0)\right]M^2_\pi\,\overset{o}{M}{}^2_\eta\\ & -\left[16L^r_4(\mu)+\frac{8}{3}L^r_5(\mu)+\frac{3}{2}J^r_{KK}(0)\right]M^2_\eta\,\overset{o}{M}{}^2_\pi\\ & -\left[\frac{8}{3}L^r_5(\mu)-\frac{1}{6}J^r_{KK}(0)+\frac{2}{3}J^r_{\pi\pi}(0)\right]M^2_\pi\,\overset{o}{M}{}^2_\pi\\ & +\left[40L^r_6(\mu)+\frac{5}{18}J^r_{\eta\eta}(0)+\frac{5}{4}J^r_{KK}(0)\right]\overset{o}{M}{}^2_\pi\overset{o}{M}{}^2_\eta\\ & +32L^r_7(\mu)\,\overset{o}{M}{}^2_\pi\,(\overset{o}{M}{}^2_\pi-\overset{o}{M}{}^2_\eta)\\ & +\left[8L^r_6(\mu)+\frac{64}{3}L^r_8(\mu)+J^r_{\pi\pi}(0)+\frac{2}{9}J^r_{\pi\eta}(0)-\frac{1}{18}J^r_{\eta\eta}(0)\right.\\ & \left.+\frac{1}{4}J^r_{KK}(0)\right]\overset{o}{M}{}^4_\pi+\frac{1}{3}F^2_\pi M^2_\pi\delta_\alpha\,, \end{aligned} \tag{90}$$

$$\begin{aligned} \beta = {} & -2\Sigma_{\eta\pi}\left[8(L^r_1(\mu)+\frac{1}{6}L^r_3(\mu))+\frac{3}{8}J^r_{KK}(0)\right]\\ & +\left[8L^r_4(\mu)\left(\overset{o}{M}{}^2_\eta+\overset{o}{M}{}^2_\pi\right)+\frac{1}{2}J^r_{KK}(0)\,\overset{o}{M}{}^2_\eta\right.\\ & \left.+\frac{1}{3}\left(J^r_{\pi\pi}(0)+\frac{1}{2}J^r_{KK}(0)\right)\overset{o}{M}{}^2_\pi\right]+\beta\delta_\beta\,, \end{aligned} \tag{91}$$

$$\begin{aligned} \gamma = {} & \left[8\left(L^r_1(\mu)+\frac{1}{6}L^r_3(\mu)\right)+\frac{3}{8}J^r_{KK}(0)\right]\\ & +\left[2\left(L^r_2(\mu)+\frac{1}{3}L^r_3(\mu)\right)+\frac{3}{16}J^r_{KK}(0)\right]+\gamma\delta_\gamma\,, \end{aligned} \tag{92}$$

$$\omega = \left[2\left(L^r_2(\mu)+\frac{1}{3}L^r_3(\mu)\right)+\frac{3}{16}J^r_{KK}(0)\right]+\omega\delta_\omega\,. \tag{93}$$

5 Reparametrization of the bare expansion

Let us now discuss the various possibilities of the reparametrization of the bare expansion.

5.1 $\pi\eta$ scattering within the standard chiral perturbation theory to $O(p^4)$

The standard way of dealing with the chiral expansion consists of two "dangerous" steps. The first one involves using the inverted expansions of the type (26) in order to express the amplitude in terms of the masses and decay constants instead of the parameters $B_0\widehat{m}$, F_0 and $r = m_s/\widehat{m}$ of the $O(p^2)$ chiral Lagrangian. Here one encounters an ambiguity connected with different possibilities of how to choose the observable G in (26), the chiral expansion of which starts with the desired $O(p^2)$ parameter G_0.

Let us fix this ambiguity by using the expansions of F^2_π, M^2_π and M^2_K, inverting of which leads to[14]

[14] Instead of M^2_K we could use the chiral expansion of M^2_η to obtain

$$r = \tilde{r}_2 = \frac{3}{2}\left(\frac{M^2_\eta}{M^2_\pi}-\frac{1}{3}\right)$$

or even $F^2_K M^2_K$ to get

$$r = r^*_2 = 2\frac{F^2_K M^2_K}{F^2_\pi M^2_\pi}-1\,.$$

The latter choice, formally as good as the previous two, could also involve the redefinition of the loop masses into $\overset{o}{M}{}^2_P=$

$$F_0^2 = F_\pi^2(1+4\mu_\pi+2\mu_K) - 8M_\pi^2\left(L_4^r(\mu)(2+r)+L_5^r(\mu)\right) , \tag{94}$$

$$2B_0\widehat{m} = M_\pi^2\Big[1-\mu_\pi+\frac{1}{3}\mu_\eta-\frac{8M_\pi^2}{F_\pi^2}\big(2L_8^r(\mu) + 2(2+r)L_6^r(\mu)-L_5^r(\mu)-(2+r)L_4^r(\mu)\big)\Big] , \tag{95}$$

$$r = r_2 = \frac{2M_K^2}{M_\pi^2} - 1 + O(p^2) . \tag{96}$$

Inserting the inverted expansions (94)–(96) into (90) and (91) and keeping terms up to the order $O(p^4)$ we get

$$\begin{aligned}\alpha = {}& \frac{1}{3}F_\pi^2M_\pi^2 + \frac{16}{3}M_\pi^2M_\eta^2L_3^r(\mu) - \frac{64}{3}L_7^r(\mu)M_\pi^4(r_2-1) \\ &+ M_\pi^2M_\eta^2\left[32L_1^r(\mu)-16L_4^r(\mu)-\frac{8}{3}L_5^r(\mu)\right] \\ &+\frac{1}{3}(2r_2+1)M_\pi^4\left[32L_6^r(\mu)-16L_4^r(\mu)+\frac{2}{9}J_{\eta\eta}^r(0)\right] \\ &+ M_\pi^4\Big[-\frac{8}{3}L_5^r(\mu)+16L_8^r(\mu)-\frac{1}{6}J_{\pi\pi}^r(0)+\frac{2}{9}J_{\pi\eta}^r(0) \\ &-\frac{1}{18}J_{\eta\eta}^r(0)+\frac{1}{3}J_{KK}^r(0)\Big] + \alpha\delta_\alpha^{\rm st} ,\end{aligned} \tag{97}$$

$$\begin{aligned}\beta = {}& -2\Sigma_{\eta\pi}\left[8\left(L_1^r(\mu)+\frac{1}{6}L_3^r(\mu)\right)+\frac{3}{8}J_{KK}^r(0)\right] \\ &+\frac{1}{3}M_\pi^2\left[16L_4^r(\mu)(r_2+2)+J_{KK}^r(0)(r_2+1)+J_{\pi\pi}^r(0)\right] \\ &+\beta\delta_\beta^{\rm st} ,\end{aligned} \tag{98}$$

with the new remainders $\delta_\alpha^{\rm st}$ and $\delta_\beta^{\rm st}$, which might be, however, out of control as we have already discussed. In fact, this first step involves three "unsafe" manipulations from the point of view of resummed χPT: using "dangerous" expansions for the masses as a starting point, the inversion and finally the negligence of all higher order terms generated by this procedure after the insertion.

Use of physical masses inside the chiral logarithms is understood. Higher order LECs are then fitted by using additional experimental input; no parameters are therefore left free. Also note that (96) effectively implements the classical Gell-Mann–Okubo formula:

$$3M_\eta^2 - 4M_K^2 + M_\pi^2 = 0 . \tag{99}$$

This insures renormalization scale independence. We, however, leave M_η at its physical value in cases when it was produced by on-shell mass on the outer legs or inside chiral logarithms, which is compatible with the requirement of scale independence.

$F_P^2M_P^2/F_\pi^2$ instead of the simple form $\overset{o}{M}{}_P^2 = M_P^2$ as in the case of the other standard reparametrizations. Even then, however, it suffers from numerically large $O(p^4)$ corrections, which could produce instabilities of the reparametrization based on this observable.

The second step is connected to the fact that the amplitude is used in standard χPT rather than $G(s,t;u)$. As was shown in Sect. 3.2, the expansion of the amplitude can be organized in various ways, of which only (54) is considered safe in the resummed approach. On the other hand, from the standard point of view it often seems more advantageous to use (56) and (57), as together with (94) it leads to only the experimentally very well known pion decay constant being present in the formulae. This can be seen in the case of F_η, which is experimentally poorly known due to η–η' mixing [43], and thus, if it is kept at its physical value as was done in [38], a significant uncertainty is introduced into the results. As the normalization (56) and (57) is used more often in NLO SχPT, we will adhere to this view and perform this second step by expanding the kaon and eta decay constants from the denominators and subsequently cutting off the higher orders.

Using therefore the prescription (60) and (61), the dispersive part of the $O(p^4)$ amplitude (81) simplifies using the reparametrization recipe described above:

$$\begin{aligned}\phi(s) = {}& \frac{1}{9}M_\pi^4\overline{J}_{\pi\eta}(s)+\frac{3}{8} \\ &\times\left[\left(s-\frac{1}{3}M_\eta^2-\frac{1}{3}M_\pi^2-\frac{2}{3}M_K^2\right)-\frac{1}{9}M_\pi^2(r_2-1)\right]^2 \\ &\times\overline{J}_{KK}(s) , \\ \phi^T(s) = {}& \frac{1}{3}M_\pi^2\left(s-\frac{1}{2}M_\pi^2\right)\overline{J}_{\pi\pi}(s)+\frac{1}{54}M_\pi^4(8r_2+1)\overline{J}_{\eta\eta}(s) \\ &+\frac{1}{8}\left[\left(s-\frac{2}{3}M_\pi^2-\frac{2}{3}M_K^2\right)+\frac{1}{3}M_\pi^2(r_2+3)\right] \\ &\times\left[\left(3s-2M_K^2-2M_\eta^2\right)+\frac{1}{3}M_\pi^2(3r_2+1)\right]\overline{J}_{KK}(s) .\end{aligned} \tag{100}$$

The second step also propagates itself to the case of the subthreshold parameters c_{ij} and the scattering lengths a_i, where it consists of the expansion of F_η^2 in the denominator of (83) and (82). This step could in principle produce an uncontrollable contribution to the remainders as well.

5.2 Resummation of the vacuum fluctuation

In order to preserve global convergence, as was discussed, in the context of resummed χPT we are not allowed to perform "dangerous" inverted expansions and thus to express the $O(p^2)$ masses $\overset{o}{M}_P$ and the decay constant F_0 in terms of the physical ones in the way it is common within the standard χPT calculations sketched above. Instead of this, the $O(p^2)$ LECs are left free, or more precisely, rewritten using parameters directly related to the order parameters of the chiral symmetry breaking:[15]

$$r = \frac{m_s}{\widehat{m}} , \quad X = \frac{\overset{o}{M}{}_\pi^2 F_0^2}{M_\pi^2F_\pi^2} , \quad Z = \frac{F_0^2}{F_\pi^2} . \tag{101}$$

[15] Here we omit the explicit dependence of X, Y and Z on N_f keeping in mind that $N_f = 3$ in what follows.

The bare expansions for the masses $F_P^2 M_P^2$ and decay constants F_P^2 are used to reparametrize the NLO LECs L_4–L_8. As the dependence is linear, this can be done in a purely non-perturbative algebraic way by introduction of an unknown higher order remainder to each observable used. The relevant formulae for L_4–L_8 can be found in Appendix E.

As the masses and decay constants do not depend on L_1, , L_2 and L_3, bare expansions of some additional, experimentally well known observables is needed for these LECs. This is, however, even if highly desirable, out of the scope of our article, and we make a shortcut and use the standard tabular values for these constants. We will make an analysis of the sensitivity of our results to a change in the value of L_1–L_3 in the next section, devoted to numerical results.

For the resulting expression for the parameters α and β, we use the following abbreviation for some repeatedly occurring combinations:

$$r_2^* = 2\frac{F_K^2 M_K^2}{F_\pi^2 M_\pi^2} - 1\,, \tag{102}$$

$$\varepsilon(r) = 2\frac{r_2^* - r}{r^2 - 1}\,, \tag{103}$$

$$\eta(r) = \frac{2}{r-1}\left(\frac{F_K^2}{F_\pi^2} - 1\right)\,, \tag{104}$$

$$\Delta_{\mathrm{GMO}} = \frac{3F_\eta^2 M_\eta^2 + F_\pi^2 M_\pi^2 - 4F_K^2 M_K^2}{F_\pi^2 M_\pi^2}\,, \tag{105}$$

in terms of which we get

$$\begin{aligned}
\alpha = {} & \frac{1}{3} X F_\pi^2 M_\pi^2 + \frac{1}{3} F_\pi^2 M_\pi^2 (1-X)\frac{5r+4}{r+2} \\
& + \frac{1}{3} F_\pi^2 M_\pi^2 \varepsilon(r) r \frac{2r+1}{r+2} - \frac{2}{3}\frac{F_\pi^2 M_\pi^2}{r-1}\Delta_{\mathrm{GMO}} \\
& + 2\frac{F_\pi^2}{r+2}(Z-1)\left(\frac{1}{3}M_\pi^2(2r+1) + M_\eta^2\right) \\
& + \frac{F_\pi^2}{r+2}\eta(r)\left(rM_\pi^2 - \frac{1}{3}(r-4)M_\eta^2\right) \\
& + \frac{1}{96\pi^2}\frac{X}{Z}M_\pi^4(4r+5) + \frac{3}{32\pi^2}\frac{X}{Z}M_\pi^2 M_\eta^2 \\
& - \frac{1}{864\pi^2}\left(\frac{X}{Z}\right)^2 M_\pi^4(44r+67) \\
& - \frac{M_\pi^4}{2(r+2)(r-1)}\frac{X}{Z}\Big[J_{\eta\eta}^r(0)(2r+1) + 2J_{KK}^r(0)r \\
& - J_{\pi\pi}^r(0)(4r+1)\Big] r \\
& + \frac{M_\pi^2 M_\eta^2}{6(r+2)(r-1)}\frac{X}{Z}\Big[J_{\eta\eta}^r(0)(2r+1)(r-4) \\
& + J_{\pi\pi}^r(0)(19r-4) - 2J_{KK}^r(0)(r^2+6r-4)\Big] \\
& + \frac{M_\pi^4}{18(r+2)(r-1)}\left(\frac{X}{Z}\right)^2\Big[J_{\eta\eta}^r(0)(5r^2-10r-4) \\
& + 6J_{KK}^r(0)(3r^2-2r-4) + 4J_{\pi\eta}^r(0)(r^2+r-2) \\
& - 9J_{\pi\pi}^r(0)(3r^2-2r-4)\Big] \\
& + 4\left[8\left(L_1^r(\mu) + \frac{1}{6}L_3^r(\mu)\right) + \frac{3}{8}J_{KK}^r(0)\right] M_\pi^2 M_\eta^2 \\
& + \frac{1}{3}F_\pi^2 M_\pi^2 \delta'_\alpha\,,
\end{aligned} \tag{106}$$

$$\begin{aligned}
\beta = {} & \frac{2}{3}F_\pi^2(1 - Z - \eta(r)) \\
& + \frac{1}{3}\frac{M_\pi^2}{r-1}\frac{X}{Z}\Big[J_{\eta\eta}^r(0)(2r+1) + J_{KK}^r(0)(r+1) \\
& - J_{\pi\pi}^r(0)(3r+2) - \frac{1}{16\pi^2}(r+2)(r-1)\Big] \\
& - 2\Sigma_{\eta\pi}[8\left(L_1^r(\mu) + \frac{1}{6}L_3^r(\mu)\right) + \frac{3}{8}J_{KK}^r(0)] + \beta\delta'_\beta\,.
\end{aligned} \tag{107}$$

The new primed remainders are the following functions of the original remainders entering the game:

$$\begin{aligned}
\delta'_\alpha = {} & \delta_\alpha - \frac{5r+4}{r+2}\delta_{F_\pi M_\pi} + \frac{2}{r+2}\left((2r+1) + \frac{3M_\eta^2}{M_\pi^2}\right)\delta_{F_\pi} \\
& - \frac{2}{(r+2)(r-1)}\left(3r - (r-4)\frac{M_\eta^2}{M_\pi^2}\right)\left(\frac{F_K^2}{F_\pi^2}\delta_{F_K} - \delta_{F_\pi}\right) \\
& - \frac{2r(2r+1)}{(r+2)(r^2-1)}\left(2\frac{F_K^2 M_K^2}{F_\pi^2 M_\pi^2}\delta_{F_K M_K} - (r+1)\delta_{F_\pi M_\pi}\right) \\
& + \frac{2}{r-1}\Bigg(3\frac{F_\eta^2 M_\eta^2}{F_\pi^2 M_\pi^2}\delta_{F_\eta M_\eta} + \delta_{F_\pi M_\pi} \\
& - 4\frac{F_K^2 M_K^2}{F_\pi^2 M_\pi^2}\delta_{F_K M_K}\Bigg)\,,
\end{aligned} \tag{108}$$

$$\delta'_\beta = \delta_\beta + \frac{2}{3}\frac{2F_K^2\delta_{F_K} - (r+1)F_\pi^2\delta_{F_\pi}}{\beta(r-1)}\,. \tag{109}$$

This is an alternative to the first step used in the standard approach to χPT. Because the value of F_η is not very well known, we make in a sense a step parallel to the second step as well, i.e. using the chiral expansion for F_η^2 in the denominator of (81)–(83). It involves the reparametrization in terms of X, Z and r (see Appendix B for details), but, contrary to the standard case, the denominator is not further expanded and the result is given in a non-perturbative resummed form of the ratio of two "safe" expansions.

5.3 $\pi\eta$ scattering within the generalized chiral perturbation theory to $O(p^4)$ – the bare expansion of $G(s,t;u)$

In analogy with (85), the strict chiral expansion for $G(s,t;u)$ within the generalized χPT can be straightforwardly obtained by using the Lagrangian summarized in Appendix F, where we use the traditional notation for the LECs. The result has the following structure:

$$G_{\pi\eta} = \tilde{G}^{(2)} + \tilde{G}^{(3)} + \tilde{G}_{\mathrm{ct}}^{(4)} + \tilde{G}_{\mathrm{tad}}^{(4)} + \tilde{G}_{\mathrm{unit}}^{(4)} + G\delta_G^{\mathrm{G}\chi\mathrm{PT}}\,, \tag{110}$$

where[16]

$$\begin{aligned}
\tilde{G}^{(2)}(s,t;u) &= \frac{1}{3}F_0^2\big[\tilde{M}_\pi^2+4\widehat{m}^2\big(3A_0-4(r-1)Z_0^P \\
&\quad +2(2r+1)Z_0^S\big)\big]\,,\\
\tilde{G}^{(3)}(s,t;u) &= \frac{1}{3}F_0^2\,\big[-2\widehat{m}\,\big(6M_\eta^2+M_\pi^2(2+4r)-2(2+r)t\big)\\
&\quad \times\tilde{\xi}-2\widehat{m}\Sigma_{\pi\eta}\xi\\
&\quad +81\widehat{m}^3\rho_1+\widehat{m}^3\rho_2+(80-64r-16r^2)\widehat{m}^3\rho_3\\
&\quad +(100+64r+34r^2)\widehat{m}^3\rho_4+(2+r^2)\widehat{m}^3\rho_5\\
&\quad +(96-96r)\widehat{m}^3\rho_6+(144+288r+108r^2)\\
&\quad \times\widehat{m}^3\rho_7\big]\,,\\
\tilde{G}^{(4)}_{\rm ct}(s,t;u) &= 8(L_1+\frac{1}{6}L_3)\,\big(t-2M_\pi^2\big)\,\big(t-2M_\eta^2\big)\\
&\quad +4\left(L_2+\frac{1}{3}L_3\right)\\
&\quad \times\left[\big(s-M_\pi^2-M_\eta^2\big)^2+\big(u-M_\pi^2-M_\eta^2\big)^2\right]\\
&\quad +\frac{8}{3}\widehat{m}^2F_0^2\Big\{-(B_1-B_2)\Sigma_{\pi\eta}\\
&\quad +2D^PM_\pi^2(r-1)-2C_1^PM_\eta^2(r-1)\\
&\quad +C_1^S(2r+1)t\\
&\quad -D^S\left[\frac{1}{2}\Sigma_{\pi\eta}(5r+4)-(2r+1)t\right]\\
&\quad -2B_4\left[3M_\eta^2+M_\pi^2(2r^2+1)-(r^2+2)t\right]\Big\}\\
&\quad +\frac{1}{3}\widehat{m}^4F_0^2\big[256E_1+16E_2\\
&\quad +F_1^P(256-256r^2)+F_4^S(32+16r^2)\\
&\quad +F_1^S(256+320r^2)\\
&\quad +F_5^{SP}(192-320r+160r^2-32r^3)\\
&\quad +F_2^P(240-216r-24r^3)\\
&\quad +F_6^{SP}(32-32r+16r^2-16r^3)\\
&\quad +F_3^P(16-8r-8r^3)+F_3^S(16+10r+10r^3)\\
&\quad +F_6^{SS}(32+40r+16r^2+20r^3)\\
&\quad +F_7^{SP}(384-160r-256r^2+32r^3)\\
&\quad +F_2^S(400+234r+74r^3)\\
&\quad +F_5^{SS}(576+720r+480r^2+168r^3)\big]\,,\\
\tilde{G}^{(4)}_{\rm tad}(s,t;u) &= -\frac{1}{9}F_0^2\big[2\widehat{m}B_0(\mu_\eta+6\mu_K+9\mu_\pi)\\
&\quad +8A_0\widehat{m}^2(8\mu_\eta+3\mu_K(r+8)+48\mu_\pi)\\
&\quad +4Z_0^S\widehat{m}^2(\mu_\eta(16+41r)+\mu_K(48+90r)\\
&\quad +\mu_\pi(96+45r))-16Z_0^P\widehat{m}^2(2\mu_\eta(5r-2)\\
&\quad +3\mu_K(6r-4)+3\mu_\pi(3r-8))\big]\,,\\
\tilde{G}^{(4)}_{\rm unit}(s,t;u) &= \frac{1}{9}\big[\tilde{M}_\pi^2+4\widehat{m}^2\big(3A_0-4(r-1)Z_0^P\\
&\quad +2(2r+1)Z_0^S\big)\big]^2\,\big[J^r_{\pi\eta}(s)+J^r_{\pi\eta}(u)\big]\\
&\quad +\frac{3}{8}\big[s-M_\pi^2-M_\eta^2+\frac{2}{3}\tilde{M}_\pi^2-\frac{8}{3}(r-1)\\
&\quad \times\widehat{m}^2\,\big(A_0+2Z_0^P\big)\,\big]^2J^r_{KK}(s)\\
&\quad +\frac{3}{8}\big[u-M_\pi^2-M_\eta^2+\frac{2}{3}\tilde{M}_\pi^2-\frac{8}{3}(r-1)\\
&\quad \times\widehat{m}^2\,\big(A_0+2Z_0^P\big)\,\big]^2J^r_{KK}(u)\\
&\quad +\frac{1}{3}\big[\tilde{M}_\pi^2+4\widehat{m}^2\big(3A_0-4(r-1)Z_0^P\\
&\quad +2(2r+1)Z_0^S\big)\big]\big[t-2M_\pi^2+\frac{3}{2}\tilde{M}_\pi^2\\
&\quad +10\widehat{m}^2\,\big(A_0+2Z_0^S\big)\,\big]J^r_{\pi\pi}(t)\\
&\quad +\frac{2}{9}\big[\tilde{M}_\pi^2+4\widehat{m}^2\big(3A_0-4(r-1)Z_0^P\\
&\quad +2(2r+1)Z_0^S\big)\big]\big[\tilde{M}_\eta^2-\frac{1}{4}\tilde{M}_\pi^2\\
&\quad +\widehat{m}^2\big((8r^2+1)A_0+8r(r-1)Z_0^P\\
&\quad +2(2r+1)^2Z_0^S\big)\big]J^r_{\eta\eta}(t)+\frac{1}{8}\big[t-2M_\pi^2\\
&\quad +2\tilde{M}_\pi^2+8(r+1)\widehat{m}^2\,\big(A_0+2Z_0^S\big)\,\big]\\
&\quad \times\Big[3t-6M_\eta^2+6\tilde{M}_\eta^2-\frac{8}{3}\tilde{M}_K^2\\
&\quad +\frac{8}{3}(r+1)\widehat{m}^2\big(3rA_0+2(r-1)Z_0^P\\
&\quad +2(2r+1)Z_0^S\big)\Big]J^r_{KK}(t)\,. \qquad (111)
\end{aligned}$$

[16] All the LECs in the following formulae are the renormalized LECs at scale μ. We have omitted the explicit notation of this in order to simplify the expressions.

In the above formulae, the generalized $O(p^2)$ masses (also present implicitly in the chiral logs μ_P and the loop functions $J^r_{PQ}(s)$) are

$$\begin{aligned}
\tilde{M}_\pi^2 &= 2\,\big[B_0+2\widehat{m}(r+2)Z_0^S\big]\,\widehat{m}+4A_0\widehat{m}^2\,,\\
\tilde{M}_K^2 &= \big[B_0+2\widehat{m}(r+2)Z_0^S\big]\,\widehat{m}(r+1)+A_0\widehat{m}^2(r+1)^2\,,\\
\tilde{M}_\eta^2 &= \frac{2}{3}\,\big[B_0+2\widehat{m}(r+2)Z_0^S\big]\,\widehat{m}(2r+1)+\frac{4}{3}A_0\widehat{m}^2(2r^2+1)\\
&\quad +\frac{8}{3}Z_0^P\widehat{m}^2(r-1)^2\,. \qquad (112)
\end{aligned}$$

The unitarity part can be further split into a polynomial and dispersive part:

$$\begin{aligned}
\tilde{G}^{(4)}_{\rm unit} &= \tilde{G}^{(4)}_{\rm unit,\,pol}+\tilde{G}^{(4)}_{\rm unit,\,disp}\\
&= \tilde{G}^{(4)}_{\rm unit}|_{J^r\to J(0)}+\tilde{G}^{(4)}_{\rm unit}|_{J^r\to\bar{J}}\,. \qquad (113)
\end{aligned}$$

According to the general prescription, the dispersive part can be replaced with that of the dispersive representation, which has the general form

$$\tilde{\mathcal{G}}^{(4)}_{\rm unit}=\phi^T(t)+\phi(s)+\phi(u)\,, \qquad (114)$$

where now

$$\begin{aligned}
\phi(s) &= F_0^4\Big\{\left(\frac{1}{3}\alpha_{\pi\eta}\tilde{M}_\pi^2\right)^2\frac{\overline{J}_{\pi\eta}(s)}{F_\pi^2F_\eta^2}\\
&\quad +\frac{3}{8}\Big[s-\frac{1}{3}M_\eta^2-\frac{1}{3}M_\pi^2-\frac{2}{3}M_K^2-\frac{1}{3}
\end{aligned}$$

$$\times\left(2\tilde{M}_K^2-\tilde{M}_\pi^2-\tilde{M}_\eta^2+\alpha_{\pi\eta K}\tilde{M}_\pi^2\right)\Bigg]^2\frac{\overline{J}_{KK}(s)}{F_K^4}\Bigg\} \tag{115}$$

$$\begin{aligned}\phi^T(s)=F_0^4\Bigg\{&\frac{1}{3}\alpha_{\pi\eta}\tilde{M}_\pi^2\left[s-\frac{4}{3}M_\pi^2+\frac{5}{6}\alpha_{\pi\pi}\tilde{M}_\pi^2\right]\frac{\overline{J}_{\pi\pi}(s)}{F_\pi^4}\\&-\frac{1}{18}\alpha_{\eta\eta}\alpha_{\pi\eta}\tilde{M}_\pi^2\left(\tilde{M}_\pi^2-4\tilde{M}_\eta^2\right)\frac{\overline{J}_{\eta\eta}(s)}{F_\eta^4}\\&+\frac{1}{8}\Bigg[s-\frac{2}{3}M_\pi^2-\frac{2}{3}M_K^2\\&+\frac{2}{3}\left(\left(\tilde{M}_K-\tilde{M}_\pi\right)^2+2\alpha_{\pi K}\tilde{M}_K\tilde{M}_\pi\right)\Bigg]\\&\times\left[3s-2M_K^2-2M_\eta^2+\alpha_{\eta K}\left(2\tilde{M}_\eta^2-\frac{2}{3}\tilde{M}_K^2\right)\right]\\&\times\frac{\overline{J}_{KK}(s)}{F_K^4}\Bigg\}\,, \end{aligned} \tag{116}$$

for (58) and (59) and analogously for (60) and (61). The coefficients $\alpha_{\pi\eta}\ldots$ parametrize the difference between the standard and the generalized cases, and within the standard $O(p^4)$ chiral expansion their values are either 1 or 0. The dependence of these constants on the LECs are given in Appendix G.

5.4 Observables of the $\pi\eta$ scattering within GχPT to $O(p^4)$ – the reparametrization

As can easily be seen from the above formulae (in fact, it is a consequence of the construction of GχPT), after identifying the parameters of the Lagrangians,

$$\begin{aligned}\frac{B_0\widehat{m}}{F_0^2}L_4^r(\mu)&\to\frac{1}{8}\widehat{m}\tilde{\xi}\,,\\ \frac{B_0\widehat{m}}{F_0^2}L_5^r(\mu)&\to\frac{1}{8}\widehat{m}\xi^r\,,\\ \frac{B_0^2\widehat{m}^2}{F_0^2}L_6^r(\mu)&\to\frac{1}{16}\widehat{m}^2Z_0^S\,,\\ \frac{B_0^2\widehat{m}^2}{F_0^2}L_7^r(\mu)&\to\frac{1}{16}\widehat{m}^2Z_0^P\,,\\ \frac{B_0^2\widehat{m}^2}{F_0^2}L_8^r(\mu)&\to\frac{1}{16}\widehat{m}^2A_0\,,\end{aligned} \tag{117}$$

and defining the remainders using the *physical* masses inside the chiral logarithms and the loop functions J_{PQ}^r, the generalized *bare* chiral expansions contain all the terms of the standard one. More precisely, the generalized $O(p^4)$ bare expansions include extra $O(p^4)$ terms, which are counted as $O(p^6)$ and $O(p^8)$ within the standard chiral power counting scheme. As a consequence, after writing the generalized bare chiral expansion of a generic "good" observable g in the form

$$g=g^{(2),\mathrm{G}\chi\mathrm{PT}}+g^{(3),\mathrm{G}\chi\mathrm{PT}}+g^{(4),\mathrm{G}\chi\mathrm{PT}}+g\delta_g^{\mathrm{G}\chi\mathrm{PT}} \tag{118}$$

and then collecting the "standard" terms together, this expansion can be formally rewritten as

$$g=g^{(2),\mathrm{std}}+g^{(4),\mathrm{std}}+g\delta_g\,,\quad g\delta_g=g\delta_g^{(G)}+g\delta_g^{\mathrm{G}\chi\mathrm{PT}}\,, \tag{119}$$

where the identification (117) is assumed. The extra $O(p^4)$ terms mentioned above are now accumulated in $g\delta_g^{(G)}$. In the case of the polynomial parameters $\alpha\ldots\omega$ (90)–(93), the two versions of the chiral expansion coincide for γ and ω:[17]

$$\delta_\gamma=\delta_\gamma^{\mathrm{G}\chi\mathrm{PT}}\,,\quad\delta_\omega=\delta_\omega^{\mathrm{G}\chi\mathrm{PT}}\,, \tag{120}$$

while the "standard" remainders δ_α and δ_β can be split into an explicitly known part, which includes the extra "non-standard" terms, and the unknown remainders inherent to GχPT:

$$\delta_\alpha=\delta_\alpha^{\mathrm{loops}}(\mu)+3\frac{\widehat{m}^2F_0^2}{F_\pi^2M_\pi^2}\delta_\alpha^{\mathrm{CT}}(\mu)+\delta_\alpha^{\mathrm{G}\chi\mathrm{PT}}\,, \tag{121}$$

$$\beta\delta_\beta=\beta\delta_\beta^{\mathrm{loops}}(\mu)+\widehat{m}^2F_0^2\delta_\beta^{\mathrm{CT}}(\mu)+\beta\delta_\beta^{\mathrm{G}\chi\mathrm{PT}}\,. \tag{122}$$

Here the first terms correspond to the new loops and the second terms to the new counterterm contributions. The explicit expressions for them can easily be extracted from the formulae of the previous subsection, the results are, however, rather lengthy and we postpone them to Appendix H.

Let us note that both $\delta_{\alpha,\beta}^{\mathrm{loop}}(\mu)$ and $\delta_{\alpha,\beta}^{\mathrm{CT}}(\mu)$ are generally renormalization scale dependent. However, due to the running of the GχPT LECs A_0, Z_0^S, Z_0^P, ξ and $\tilde{\xi}$, which, after the identification (117), is the same as in the standard case, the "standard" remainders δ_α and δ_β, given by (121) and (122), are μ independent. Of course, the "true GχPT" remainders $\delta_\alpha^{\mathrm{G}\chi\mathrm{PT}},\ldots,\delta_\omega^{\mathrm{G}\chi\mathrm{PT}}$ are scale independent by construction. That means that the sum of the loop and counterterm contributions to the "standard" remainders is μ independent too.

The usual way to handle the reparametrization of the GχPT bare expansions is quite similar to the standard one. The difference is that as there are three additional $O(p^2)$ LECs, and not all of them can be reparametrized using the inverted mass and decay constant expansions. The solution is to leave two of them free (e.g. r and $\zeta=Z_0^S/A_0$). Consequently, the expansion is performed according to the generalized power counting scheme and the terms of order higher than $O(p^4)$ are discarded.

We shall, however, not use this approach, but rather exploit the relation (119), i.e. sew the standard and generalized bare expansions together. The reparametrization is then an extension of the resummed one (Appendix E), where all the remainders of the mass and decay constant bare expansions are split according to (119). We can use all the resummed formulae as they are exact algebraic identities, valid independently on the version of χPT. The generalized contributions to the remainders can be found in

[17] The reason is that they stem from the terms quadratic in the Mandelstam variables.

Appendices B and I. The outcome for the parameters α and β is then obtained by simply inserting all the generalized results for the remainders (Appendices B, H and I) into the expression for δ'_α and δ'_β (108) and (109).

After this procedure, the generalized LECs are present only in the formulae for the standard remainders. Also note that $\delta_\alpha^{\text{loops}}(\mu)$ and $\delta_\beta^{\text{loops}}(\mu)$ as well as the generalized loop contributions to the mass and decay constant remainders depend explicitly on the $O(p^2)$ LECs $B_0 = XM_\pi^2/2\widehat{m}$, $F_0^2 = ZF_\pi^2$, A_0, Z_0^S and Z_0^P.[18] So as the last step of the reparametrization, the remaining dependence of the generic "loop" remainders $\delta_\alpha^{\text{loops}}(\mu), \ldots$, etc. on the $O(p^2)$ LECs F_0, A_0, Z_0^S and Z_0^P can be removed up to the order $O(p^4)$ using the leading order expressions

$$\begin{aligned} F_0^2 &= F_K^2 = F_\eta^2 \to F_\pi^2\,, \\ \widehat{m}^2 F_0^2 Z_0^S &\to \frac{1}{4}\frac{F_\pi^2 M_\pi^2}{r+2}(1-X-\varepsilon(r))\,, \\ \widehat{m}^2 F_0^2 Z_0^P &\to -\frac{1}{8}F_\pi^2 M_\pi^2\left(\varepsilon(r)-\frac{\Delta_{\text{GMO}}}{(r-1)^2}\right)\,, \\ \widehat{m}^2 F_0^2 A_0 &\to \frac{1}{4}F_\pi^2 M_\pi^2 \varepsilon(r)\,. \end{aligned} \tag{123}$$

As a summary, our handling of the generalized bare expansion can be viewed in two ways – either as a partial estimate of the standard remainders present in the resummed approach or as a special treatment within the generalized framework, where the $O(p^2)$ (and partly $O(p^3)$) LECs are reparametrized algebraically at the leading order, while they are treated perturbatively at the $O(p^4)$ one. The numerical results including a simple estimate of the remaining NLO and NNLO LECs are presented in Sect. 6.6; also, Appendix B contains an illustrative example of applying this procedure on F_η.

6 Numerical results

In this section we shall present the numerical analysis of the observables connected to the $\pi\eta$ scattering amplitude and the results which illustrate the subtleties of the various versions of the chiral expansions described above. In the numerical estimates we use $M_\pi = 135\,\text{MeV}$, $M_\eta = 548\,\text{MeV}$, $M_K = 496\,\text{MeV}$, $\mu = M_\rho = 770\,\text{MeV}$, $F_\pi = 92.4\,\text{MeV}$ and $F_K = 113\,\text{MeV}$. For the calculation within the standard χPT, the $O(p^4)$ LECs are taken from [44–46]. In the alternative reparametrization schemes, where only L_1, L_2 and L_3 remain among the free parameters and the other $O(p^4)$ LECs are expressed in terms of physical masses, decay constants and the indirect remainders, we again keep (though rather non-systematically) the values of L_1, L_2 and L_3 from the same references. The sensitivity on this LECs might be then estimated by means of the variation around these central values. In this chapter we insert the physical masses into the functions $J^r_{PQ}(0)$ unless stated otherwise.

6.1 The standard chiral perturbation theory

This subsection discusses the predictions of the standard chiral expansion to the order $O(p^4)$, which are summarized in Tables 1 and 2. Let us start with the parameter α of the polynomial part of the amplitude. The relevant formulae from Sect. 5.1 and the LECs taken from [44][19] result in the following value:

$$\alpha = \frac{1}{3}\left(1+0.683+\delta_\alpha^{\text{st}}\right)F_\pi^2 M_\pi^2\,. \tag{124}$$

In this expression, the first term corresponds to the current algebra result $\alpha^{\text{CA}} = F_\pi^2 M_\pi^2/3$, while the second one represents the $O(p^4)$ correction. The third term is the standard remainder, which might be out of control when $X, Z \ll 1$ and r far from r_2, even if the bare expansion of α were globally convergent (let us recall that α is a "good" observable) as we have discussed in Sect. 2. Let us also notice the unusually large next-to-leading correction, which could also indirectly indicate the numerical importance of the remainder in this scheme.

The actual numerical value of the NLO correction is very sensitive to a shift in the $O(p^4)$ LECs. The corresponding variation $\Delta\alpha$ is numerically

$$\begin{aligned} \frac{\Delta\alpha}{\alpha^{\text{CA}}} = {}&(3.38\Delta L_1 + 0.56\Delta L_3 - 3.50\Delta L_4 - 0.30\Delta L_5 \\ &+ 3.62\Delta L_6 - 3.42\Delta L_7 + 0.10\Delta L_8)\times 10^3\,. \end{aligned} \tag{125}$$

For example, using the $O(p^6)$ analysis based LECs from [45, 46] instead of those from [44], we get (cf. Table 1)[20]

$$\frac{\Delta\alpha}{\alpha^{\text{CA}}} = 0.23\,. \tag{126}$$

Note that the large coefficients in front of the L_4 and L_6 contributions indicate sensitivity of this observable to the vacuum fluctuations of $\bar{s}s$ pairs as mentioned in the introduction.

Let us compare this case with the related "dangerous" observable, namely the subthreshold parameter c_{00}. From (83) we get

$$c_{00} = \frac{1}{3}(1+0.683-0.625+0.006)\frac{M_\pi^2}{F_\pi^2} = 1.064 c_{00}^{\text{CA}}\,, \tag{127}$$

[18] More precisely, the loops depend on the "true $O(p^2)$ LECs" $\overline{A}_0$, $\overline{Z}_0^{S,P}$ (cf. Appendix F), the difference is however of higher order in the generalized power counting.

[19] This set of LECs is used in numerical estimates unless stated otherwise.

[20] Because the values of the LECs L_i based on the $O(p^6)$ fit implicitly include parts of the $O(p^6)$ corrections, the large variation can be interpreted as a signal of the importance of the NNLO contributions to the parameter α. The same is true for the other observables from Table 1.

Table 1. Standard $O(p^4)$ values of the polynomial parameters for the two sets of LECs taken form [44–46]. In the last row, the sensitivity on the LECs is (over)estimated by adding the uncertainties associated with the LECs [44] in quadrature (this is of course only a rough estimate, because in fact not all the uncertainties of the L_i are independent)

L_i	$\alpha/\alpha^{\mathrm{CA}}$	$10^3\beta/M_\eta^2$	$10^3\gamma$	$10^4\omega$
[44]	1.68	0.90	–1.52	2.24
[45, 46]	1.91	–0.68	–0.23	–5.03
Δ	2.48	7.49	3.31	9.48

Table 2. Standard $O(p^4)$ values of the subthreshold and threshold parameters as in Table 1. The c_{ij} parameters are given in their natural units, as described in the main text. Analogously to Table 1, Δ is the sensitivity on the LECs (over-) estimated by adding the uncertainties associated with the LECs [44] in quadrature

L_i	$c_{00}/c_{00}^{\mathrm{CA}}$	10^3c_{10}	10^3c_{20}	10^3c_{01}	a_0/a_0^{CA}	10^3a_1
[44]	1.06	0.91	–1.23	8.27	1.96	0.59
[45, 46]	1.51	–0.67	0.07	–3.36	1.18	–0.60
Δ	2.49	7.49	3.31	15.16	3.21	2.80

where the individual terms are the leading order contribution $c_{00}^{\mathrm{CA}} = M_\pi^2/3F_\pi^2$, the next-to-leading correction to the parameter α, the next-to-leading correction to F_η^2 induced by the expansion of the denominator and the contribution stemming from the unitarity correction $\phi(s)$, respectively. The first two large corrections accidentally cancel here; this, however, does not automatically imply a similar cancellation of the potentially large remainders (we have not written them down explicitly here). Also, the strong sensitivity of α to the variation of the LECs propagates here, giving

$$\frac{\Delta c_{00}}{c_{00}^{\mathrm{CA}}} = \frac{\Delta\alpha}{\alpha^{\mathrm{CA}}} - 0.28\Delta L_5 \times 10^3 \qquad (128)$$

and it furthermore increases the uncertainty of the $O(p^4)$ correction. This strong sensitivity supports the possibility that the standard remainders for c_{00} might be numerically larger than the next-to-leading correction. Namely using the LECs from the $O(p^6)$ fit [45, 46], which generates part of the $O(p^6)$ corrections to the reparametrized expansion of c_{00}, we get

$$\frac{\Delta c_{00}}{c_{00}^{\mathrm{CA}}} = 0.45\,. \qquad (129)$$

We can also check the sensitivity of the next-to-leading order contributions to the way we rewrite them in terms of the physical masses and decay constants (i.e. how we use the $O(p^2)$ relations generating here a difference of the order $O(p^6)$). Provided that we insert the alternative $O(p^2)$ expressions for r into the chiral expansions of α and c_{00}:

$$\tilde{r}_2 = \frac{1}{2}\left(\frac{3M_\eta^2}{M_\pi^2} - 1\right) = 24.2\,, \qquad (130)$$

$$r_2^* = 2\frac{F_K^2 M_K^2}{F_\pi^2 M_\pi^2} - 1 = 39.4\,, \qquad (131)$$

instead of the standard $O(p^2)$ value $r = r_2 = 2M_K^2/M_\pi^2 - 1 = 25.9$, we get as a result

$$\tilde{\alpha} = \frac{1}{3}(1 + 0.601)F_\pi^2 M_\pi^2\,, \qquad (132)$$

$$\tilde{c}_{00} = \frac{1}{3}(1 + 0.031)\frac{M_\pi^2}{F_\pi^2} \qquad (133)$$

and

$$\alpha^* = \frac{1}{3}(1 + 1.297)F_\pi^2 M_\pi^2\,, \qquad (134)$$

$$c_{00}^* = \frac{1}{3}(1 + 0.325)\frac{M_\pi^2}{F_\pi^2}\,. \qquad (135)$$

The remaining parameters of the polynomial part start at $O(p^4)$ and we get them from (98), (92) and (93). Their numerical values and the related subthreshold parameters c_{ij} in natural units (chosen in such a way as to make the comparison with the polynomial parameters easy, i.e. we take c_{10} and c_{01} in units of M_η^2/F_π^2 and c_{20} in units of F_π^{-4}, cf. (83)) are shown in Tables 1 and 2 for the two sets of $O(p^4)$ LECs.

All the considered parameters are strongly sensitive to the variations of the LECs. For instance, the parameter β varies with the L_i as

$$\Delta\beta = (17.0\Delta L_1 - 2.8\Delta L_3 + 9.1\Delta L_4)M_\eta^2\,. \qquad (136)$$

For the LECs [44] we get $\beta = 0.90 \times 10^{-3} M_\eta^2$. Using the set [45, 46] we get a drastic change

$$\Delta\beta = -1.58 \times 10^{-3} M_\eta^2\,. \qquad (137)$$

Let us turn to the "doubly dangerous" observables represented by the scattering lengths now. For the s-wave we obtain from (82) and the LECs [44]

$$\begin{aligned} a_0 &= \frac{1}{24\pi F_\pi^2}\frac{M_\pi^3}{M_\eta + M_\pi}(1 + 0.683 + 0.378 - 0.625 + 0.527) \\ &= \frac{1}{24\pi F_\pi^2}\frac{M_\pi^3}{M_\eta + M_\pi}(1 + 0.963) = 11.0 \times 10^{-3}\,. \end{aligned} \qquad (138)$$

Here the individual terms in the first line represent the current algebra result, the correction stemming from the $O(p^4)$ contributions to the parameters α and ω, the next-to-leading correction to F_η^2 induced by the expansion of the denominator and the correction induced by the dispersive part of the amplitude $\phi(s)$, in this order. This result confirms the expectations about bad convergence of the chiral expansion for the observables which are connected to the threshold values of the amplitude – even if the polynomial

NLO corrections were small, which they are not, the dispersive part would still be as large as 50% of the leading order term.

The sensitivity to the $O(p^4)$ LECs is illustrated in Table 2. The p-wave scattering length then starts at $O(p^4)$, we get the values in the last column of the table from (82).

When comparing our standard χPT results for the scattering lengths (first row of Table 2),

$$a_0 = 11.0\times 10^{-3}\,,\quad a_1 = 5.9\times 10^{-4}\,, \tag{139}$$

with those of [38], quoted in the Introduction,

$$a_0^{\mathrm{BKM}} = 7.2\times 10^{-3}\,,\quad a_1^{\mathrm{BKM}} = -5.2\times 10^{-4}\,, \tag{140}$$

we can see a seemingly large discrepancy. The difference is produced by a different set of $O(p^4)$ LECs, the alternative treatment of the F_η in the denominator and by another form of the unitarity corrections – the authors do not use a matching with a dispersive representation. Taken these distinctions into account, we get more consistent numbers (with our inputs for the masses and decay constants):

$$a_0 = 7.0\times 10^{-3}\,,\quad a_1 = -5.0\times 10^{-4}\,. \tag{141}$$

As we can see, a slightly different treatment of the standard chiral expansion may lead to a significant shift in the results. This does not necessarily mean that the standard counting is not consistent, though. As follows from from Table 2, the nominal uncertainty associated with the $O(p^4)$ LEC error bars encompasses the difference

$$\Delta a_0 = 18.0\times 10^{-3}\,,\quad \Delta a_1 = 28.0\times 10^{-4}\,. \tag{142}$$

What can be concluded is that the standard approach has a large theoretical uncertainty attached, which is hard to estimate. The sensitivity to the L_i^r values also leads to a considerable difference when one uses the $O(p^6)$ fit (second row in Table 2). As the two fits effectively differ only in a rearrangement of the expansion, both cannot have small higher order corrections at the same time.

6.2 Resummation of vacuum fluctuations – basic properties

In the resummed case, the free parameters are X, Z, and r together with the remaining LECs L_1, L_2 and L_3 and the direct and indirect remainders $\delta_\alpha \ldots \delta_\omega$ and δ_{F_P}, $\delta_{F_P M_P}$. Because F_η is experimentally not known with high enough accuracy, we also have to fix how to treat the observable Δ_{GMO}, which was introduced to eliminate the LEC L_7 using the bare expansion for $F_\eta^2 M_\eta^2$. Let us recall that our definition of Δ_{GMO} follows [8], where it is based on the good observables $F_P^2 M_P^2$ instead of M_P^2 and differs from that originally defined in [3]. One possibility is to treat Δ_{GMO} as an additional independent parameter. The other, similarly to the treatment of F_η in the denominators of (81), (82) and (83), is to use a (resummed) chiral expansion of F_η^2 inserted into Δ_{GMO} for the numerical estimates, i.e. to insert the following exact algebraic identity into (106) (cf. Appendix B for details):

$$\begin{aligned} F_\pi^2 M_\pi^2 \Delta_{\mathrm{GMO}} = {}& F_\pi^2 M_\pi^2 - 4M_K^2 F_K^2 + M_\eta^2 (1-\delta_{F_\eta})^{-1} \\ & \times \Bigg(4F_K^2(1-\delta_{F_K}) - F_\pi^2(1-\delta_{F_\pi}) \\ & - M_\pi^2 \left(\frac{X}{Z}\right) \left(J^r_{\pi\pi}(0) - 2(r+1)J^r_{KK}(0)\right. \\ & \left. + (2r+1)J^r_{\eta\eta}(0)\right)\Bigg)\,. \end{aligned} \tag{143}$$

This generates the indirect remainder δ_{F_η} in a nonlinear way.

Before doing a more detailed analysis, let us first illustrate the numerical sensitivity connected with the subtleties of the definition of the bare expansion. As we have discussed in Sect. 3, there is still some freedom on how to define the amplitudes entering the dispersive part of the $G_{\pi\eta}$ (cf. (58)–(61)) and also how to treat the masses inside the chiral logs. Based on general considerations it was argued [9] that in the latter case the different prescriptions should not make much difference. Nevertheless, it might be interesting to test this assumption numerically in our concrete case and also to check what is the numerical influence of the varying amplitude definition.

In Fig. 1 we plot a comparison of various definitions of the dispersive part of the amplitude using the scattering length a_0 as an example, i.e. we illustrate its sensitivity on the various versions of the unitarity corrections. The cusps on the full line, which uses the *strict* chiral expansion with the unphysical choice of the $O(p^2)$ masses in all J_{ij}^r, originate in the conflict of the physical masses used for the on-shell outer legs and the unphysical location of the thresholds. This illustrates the fact that the original strict chiral expansion is unsuitable for realistic physical predictions and its redefinition into a bare one is necessary. The

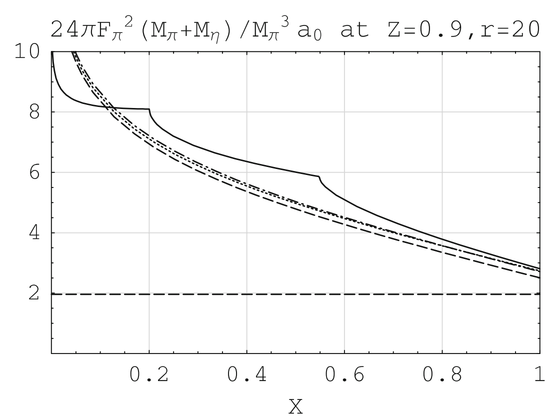

Fig. 1. Comparison of the numerical impact of the various forms of the dispersive part on the scattering length a_0. The *full line* represents the strict chiral expansion, *dotted, dashed and dash-dotted lines* the "minimal" modification, (58)–(61), respectively. The *horizontal line* shows our standard NLO prediction

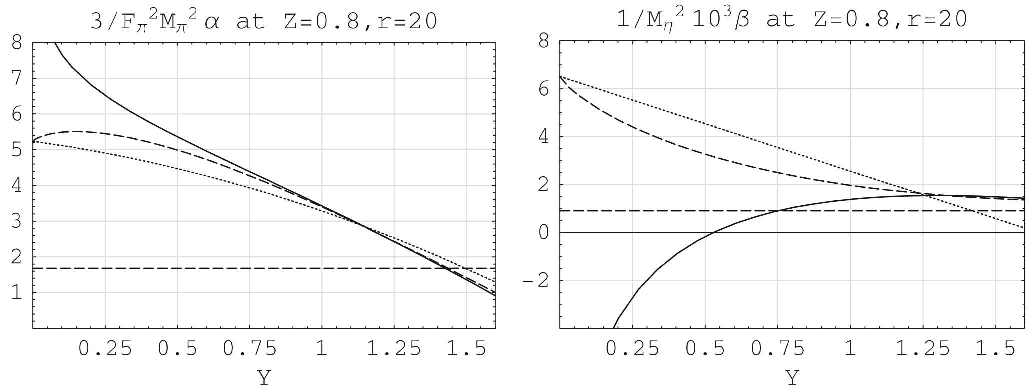

Fig. 2. In this figure we illustrate the sensitivity of the "good" variables α and β to the treatment of the chiral logs. The *full line* corresponds to the $O(p^2)$ masses in all the $J^r_{ij}(0)$, while the *dotted* and *dashed lines* correspond to the physical masses either in all $J^r_{ij}(0)$ or only in the $J^r_{ij}(0)$ originating from the unitarity corrections. *Horizontal lines* are standard NLO predictions

dotted line shows the "minimal" physical modification of the strict expansion by means of the insertion of physical masses into all J^r_{ij}. While the "minimal" version and the unitary choice (60) and (61) give numerically almost the same result, the difference between these two and the third possibility (58) and (59) is up to $\sim 0.3a_0^{\mathrm{CA}}$.

Figure 2 shows the dependence of the polynomial parameters α and β on $Y = X/Z$ for $Z = 0.8$ and $r = 20$, using the the various possible treatments of chiral logarithms in the bare expansion. The results demonstrate that the difference might be numerically important in some range of Y. For α the various possibilities do not differ drastically in comparison with the value of α itself; on the other hand the differences become comparable with α^{CA} at $Y \sim 0.5$. As we have discussed in Sect. 2, in the region of small Y the case with $O(p^2)$ masses in the tadpoles only should not differ drastically from the case when all the masses are physical. However, the convergence to the common value at $Y = 0$ is rather slow and in the intermediate region of Y the difference of these two cases for α is $\sim 0.5\alpha^{\mathrm{CA}}$ in a relatively wide interval. Keeping the $O(p^2)$ masses also in the unitarity corrections produces instabilities for $Y \to 0$, as expected. The parameter β (which starts at $O(p^4)$) is even much more sensitive.

In the following numerical analysis we take a pragmatic point of view and fix the bare expansion in such a way that the comparison of the resummed and standard reparametrizations remains as simple as possible, i.e. we insert physical masses into $J^r_{ij}(0)$ and define the amplitude according to (56), (57) and (60), (61).

6.3 Numerical comparison of the resummed and standard reparametrization

Within standard χPT we have an $O(p^4)$ prediction for X, Z, r and Δ_{GMO} based on the standard formal $O(p^4)$ chiral expansion (A.1) and (A.3); see Appendix A. Using the LECs from [44] and [45, 46], we get numerically the central values in Table 3, which should confirm the self-consistency of the standard chiral expansion scheme. As we can see, while the expectations are fulfilled in the first case, there is a considerable shift when using the $O(p^6)$ fitted constants. These numbers, moreover, should be taken with some caution, because they originate in the expansions of the "dangerous" observables and can be therefore plagued with large $O(p^6)$ remainders as well as with strong sensitivity to the $O(p^4)$ LECs.[21] In Table 3 r^{std} stems from the chiral expansion of r_2, while $r^{*\mathrm{std}}$ uses an expansion of r_2^*.[22]

Table 3. Standard values of X, Z, r and Δ_{GMO}

L_i set	X^{std}	Z^{std}	r^{std}	$r^{*\mathrm{std}}$	$\Delta^{\mathrm{std}}_{\mathrm{GMO}}$
[44]	0.902	0.865	25.2	26.7	6.41
[45, 46]	0.726	0.734	25.9	31.7	3.31

Let us now illustrate the relationship of the resummed and standard approach using the observables from Sect. 6.1.

For the "good" observables α and β we can expect that the numerical values of X^{std}, Z^{std}, r^{std} and $\Delta^{\mathrm{std}}_{\mathrm{GMO}}$, with L_1, L_2 and L_3 taken from [44] for definiteness, should produce numbers consistent with the first row of Table 1 when inserted into ((106) and (107)). The results for the various possibilities of how to approach the standard predictions for α and β (which is independent on Δ_{GMO}) within the resummed version of χPT are summarized in Table 4.[23] The last row corresponds to the resummed treatment of Δ_{GMO} explained above. The dependence of the central values of α and β on the parameters r, X and Z in the broader vicinity of their standard values is illustrated in Fig. 3. These results can be interpreted as a token of consistency of both

[21] As was analyzed in detail in [8], the actual values of X^{std} and Z^{std} are strongly sensitive to the values of the LECs L_6 and L_4 connected with the vacuum fluctuation of the $\bar{s}s$ pairs, and the same is true for the sensitivity of r^{std} and $\Delta^{\mathrm{std}}_{\mathrm{GMO}}$ to L_8 and L_7. This causes large error bars to be attached to these values. Nevertheless, in the following we take these central values as a reference point for an illustrative numerical comparison of the two versions of the chiral expansion.

[22] Though the difference between the values of r^{std} and $r^{*\mathrm{std}}$ is within the standardly expected accuracy of the $O(p^4)$ approximation, note that for $r^{*\mathrm{std}}$ the $O(p^4)$ correction is much larger than in the first alternative ($r_2 = 25.9$ while $r_2^* = 39.4$).

[23] In this and the following tables in this subsection we ignore the uncertainty stemming from the remainders and L_i, $i = 1, 2, 3$, and we give only the central values (assuming the central values of the remainders to be zero).

Table 4. The values of the polynomial parameters α and β and the related subthreshold and threshold parameters near the standard reference point. For Δ_{GMO} we take either the standard value ($\Delta_{\mathrm{GMO}}^{\mathrm{std}}$) or the resummed prediction described (Δ_{GMO}) in the main text

X	Z	r	Δ_{GMO}	$\alpha/\alpha^{\mathrm{CA}}$	$10^3\beta$	$c_{00}/c_{00}^{\mathrm{CA}}$	$10^3 c_{10}$	a_0/a_0^{CA}	$10^3 a_1$
X^{std}	Z^{std}	r^{std}	$\Delta_{\mathrm{GMO}}^{\mathrm{std}}$	1.88	0.69	1.11	0.41	1.64	0.30
X^{std}	Z^{std}	$r^{*\mathrm{std}}$	$\Delta_{\mathrm{GMO}}^{\mathrm{std}}$	1.61	0.55	0.95	0.33	1.47	0.26
X^{std}	Z^{std}	r_2	$\Delta_{\mathrm{GMO}}^{\mathrm{std}}$	1.74	0.62	1.03	0.37	1.55	0.28
Z^{std}	Z^{std}	$r^{*\mathrm{std}}$	$\Delta_{\mathrm{GMO}}^{\mathrm{std}}$	1.76	0.75	1.04	0.45	1.57	0.31
Z^{std}	Z^{std}	r^{std}	$\Delta_{\mathrm{GMO}}^{\mathrm{std}}$	2.02	0.89	1.20	0.53	1.72	0.35
X^{std}	Z^{std}	r^{std}	Δ_{GMO}	2.07	0.69	1.22	0.41	1.75	0.30
X^{std}	Z^{std}	$r^{*\mathrm{std}}$	Δ_{GMO}	1.78	0.55	1.05	0.33	1.57	0.26

variants of reparametrization for good observables near the standard reference point X^{std}, Z^{std} and r^{std}, where the predictions of the resummed version almost coincide with the standard results.[24] This coincidence together with the working hypothesis about the controllable remainders of good observables within the resummed reparametrization scheme confirms again the self-consistency of the standard expansion based on the assumption $X \sim 1$, $Z \sim 1$ and $r \sim r_2$. Away from the standard reference point, however, the standard reparametrization might be dangerous in the sense that the difference between the standard and the resummed prediction diverges rapidly and the importance of the standard $O(p^6)$ remainders might therefore increase considerably.

For the "dangerous" observables like c_{ij} we cannot a priori expect coincidence of both expansions even near the standard values of X, Z, and r due to the different treatments of the denominators, which contain large $O(p^4)$ corrections and are not expanded in the resummed case. Comparison of both approaches is illustrated in Table 4 (with the same treatments of Δ_{GMO} as above), Table 5 and in Fig. 4. For the dispersive part we use the prescription (60) and (61), which differs from the corresponding standard contributions of the unitarity corrections to c_{ij} for $X = Z = 1$ and $r = r_2$ by a factor $F_\pi^2/F_\eta^2 \approx 0.6$. This is reflected by the values of those c_{ij} that start at $O(p^4)$ (cf. Tables 4 and 5). Namely in this case the contribution of the polynomial part is reduced near the reference point roughly by the same factor with respect to the standard value (which includes only the first term of the expansion of the denominator). On the other hand, c_{00} is compatible with the standard value, because of the large $O(p^2)$ contribution, the tiny dispersive contribution and the fact that within the standard reparametrization of the bare expansion also the second term from the expansion of the denominator is taken into account.

[24] As a rule, the point X^{std}, Z^{std} and r^{std} cannot give the best coincidence with the standard values in all cases. The reason can be understood e.g. by having a closer look on the resummed reparametrization of β (cf. (107)). In order to reproduce the dependence of β on L_4 satisfactorily, we need $Z = Z^{\mathrm{std}}$ and $r = r^{\mathrm{std}}$; on the other hand to reproduce the chiral logs we rather need $X/Z = 1$ and $r = r_2$. This may explain why β approaches the standard value best for $X = Z = Z^{\mathrm{std}}$.

Table 5. The values of the subthreshold parameters c_{20} and c_{01} related to the polynomial parameters γ and ω at the standard reference point

X	Z	r	$10^4 c_{20}$	$10^3 c_{01}$
X^{std}	Z^{std}	r^{std}	-7.10	4.86
X^{std}	Z^{std}	$r^{*\mathrm{std}}$	-6.97	4.79
X^{std}	Z^{std}	r_2	-7.04	4.83

Let us now proceed to the "doubly dangerous" observables a_0 and a_1. These are related to the values of the amplitude at the threshold and receive therefore a large contribution from the dispersive part of the amplitude. While a_0 is reproduced well at the standard reference point, a_1 (which starts at the NLO) is off the standard value roughly by a factor 0.6 from the same reasons as for the c_{ij} parameters. The dependence of these observables on X, Z and r is depicted in Fig. 5.

6.4 The role of the remainders

Up to now we have not discussed the uncertainties of the observables calculated within the resummed scheme. They are connected with the direct and indirect remainders as well as with the LECs L_i, $i = 1, 2, 3$. As a first illustration, we have added the error bars stemming from the remainders to the central values of the various observables depicted in Figs. 3–5. These illustrate the rough estimate of the remainders $\delta \sim (30\%)^2 \sim 0.1$ as suggested in [8] and adding the uncertainties in quadrature.

In more detail, at the standard reference point $X = X^{\mathrm{std}}$, $Z = Z^{\mathrm{std}}$ and $r = r^{\mathrm{std}}$, using (108), (109) and (125), (136), we numerically get for the corresponding variations (to the first order in the remainders)

$$\frac{\Delta\alpha}{\alpha^{\mathrm{CA}}} = (\delta_\alpha + 6.92\delta_{F_\eta M_\eta} - 12.71\delta_{F_K M_K} - 0.76\delta_{F_\pi M_\pi} + 9.37\delta_{F_K} + 5.22\delta_{F_\pi} - 6.92\delta_{F_\eta}) + (3.38\Delta L_1 + 0.56\Delta L_3) \times 10^3 , \qquad (144)$$

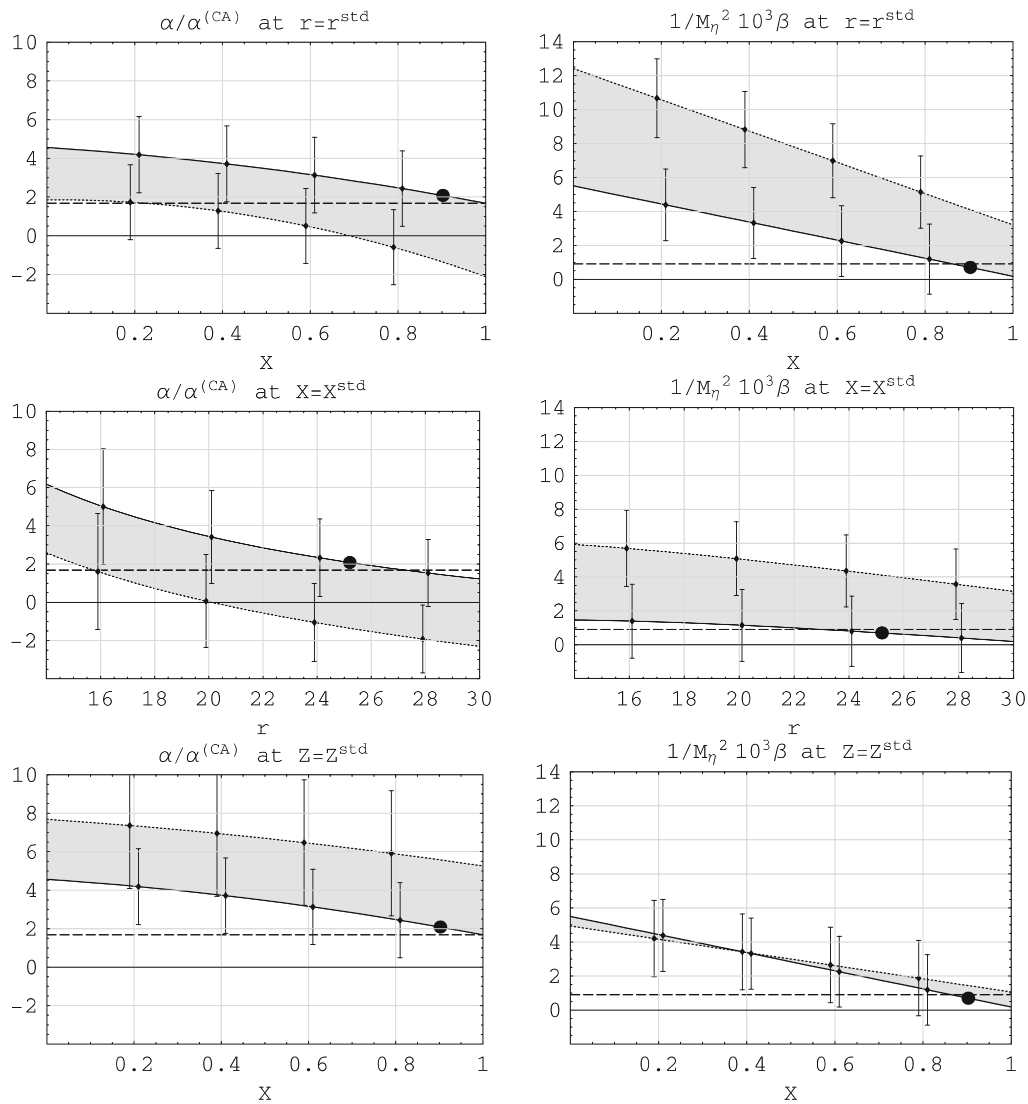

Fig. 3. The dependence of the parameters α and β on r, X and Z is plotted, one of the parameters being fixed at its standard reference value in each figure. The *dashed horizontal line* shows the standard values from the first row of the Table 1, the *full circle* depicts the corresponding resummed value at the standard reference point $[r^{\text{std}}, X^{\text{std}}, Z^{\text{std}}]$. The *error bars* represent the 10% uncertainties from the direct and indirect remainders added in quadrature. In the first row, r is fixed at r^{std}, the *filled* areas highlight the dependence on Z between $Z = Z^{\text{std}}$ (*solid line*) and $Z = 0.5$ (*dotted one*). Similarly, in the second row $X = X^{\text{std}}$, the *filled area* shows the dependence on Z again. Z is fixed at Z^{std} in the last row, the *solid line* shows the case with $r = r^{\text{std}}$, the *dotted* one the one with a lower value $r = 15$

$$\Delta\beta = \big[(0.69\delta_\beta + 2.34\delta_{F_K} - 20.52\delta_{F_\pi}) + (17.0\Delta L_1 - 2.8\Delta L_3) \times 10^3\big] \times 10^{-3} M_\eta^2 . \quad (145)$$

This reveals a strong sensitivity on both the δs and the LECs. Assuming again the typical size of the remainders to be $\delta \sim 0.1$ and adding all the uncertainties in quadrature (for ΔL_i we take the error bars from [44]) we obtain the rough (over-) estimates

$$\left|\frac{\overline{\Delta\alpha}}{\alpha^{\text{CA}}}\right| = \sqrt{1.93^2 + 1.19^2} = 2.27 , \quad (146)$$

$$|\overline{\Delta\beta}| = \sqrt{2.06^2 + 5.96^2} \times 10^{-3} M_\eta^2 = 6.31 \times 10^{-3} M_\eta^2 , \quad (147)$$

where the first number under the square root represents the contribution of the remainders, while the second one accumulates the uncertainty from $L_{1,3}$. Though these numbers are a little bit more optimistic than those in the last row of Table 1 (note that the latter originated purely from the uncertainties of ΔL_i and did not include any estimates of the higher order corrections to α and β), it is clear that, without more restrictive information on the remainders (and L_i, $i = 1, 2, 3$)[25], the predictive power of χPT is reduced considerably in the case of $\pi\eta$ scattering even for "good" observables. In other words, small remainders are not a guarantee of an equivalently small final uncertainty. In what follows, we therefore try to gain some additional information outside the (resummed) χPT expansion to get further estimates of the size of the remainders.

The sources of the remainders are twofold: on the one hand there are the unknown terms of the pure derivative expansion, on the other hand the contributions coming from the expansion in the quark masses. We try to get estimates for both of them from different sources, namely using the

[25] As already discussed, the explicit dependence on these constants could be eliminated by means of a reparametrization similar to those for L_i, $i = 4, \ldots, 6$ using further experimental input e.g. from K_{e4} decay. The price to pay is that one must introduce additional remainders connected with observables used for such a reparametrization.

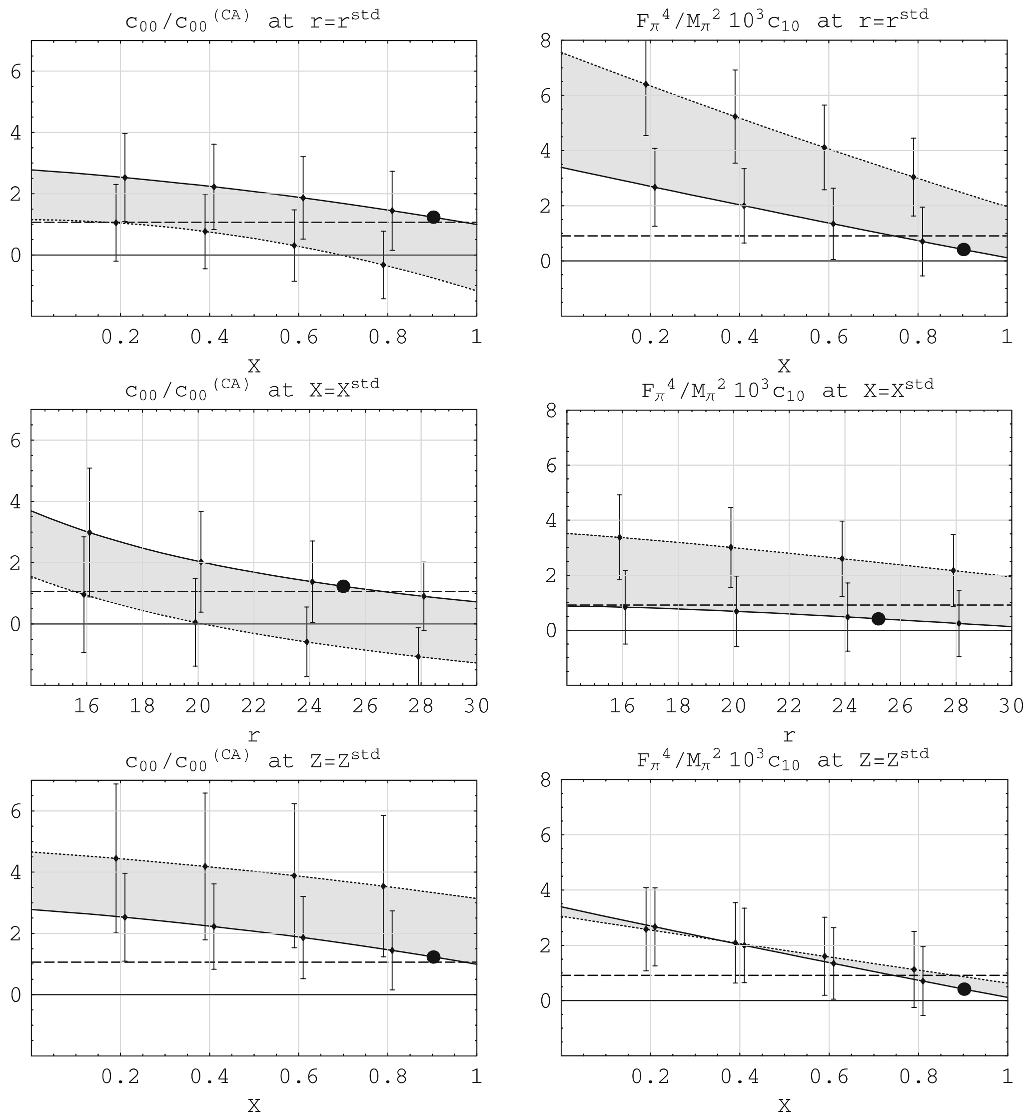

Fig. 4. Dependence of the subthreshold parameters c_{00} and c_{10} related to the polynomial parameters α and β. The figures are in one-to-one correspondence to those in Fig. 3

resonance estimate for the first type as well as independent information from generalized χPT for the second.

6.5 Resonance estimate of the direct remainders

In order to partially estimate the derivative part of the higher order corrections to the chiral expansion, we use the assumption that the process under consideration is saturated by the exchange of the lowest lying resonances, the interactions of which can be described by the Lagrangian of the resonance chiral theory (RχT). The leading order Lagrangian of RχT was originally formulated in the seminal paper [47] and applied to $\pi\eta$ scattering in [38]. To this process, only scalar resonances as well as η_8–η_0 mixing contribute. Our result for the amplitude agree with [38] (cf. Appendix J), which we can rewrite in terms of the resonance contribution $G^{\mathrm{R}}_{\pi\eta}$ to $G_{\pi\eta}$ in the form

$$\begin{aligned}G^{\mathrm{R}}_{\pi\eta}(s,t;u) = {}&\alpha^{(4)}_{\mathrm{R}} + \beta^{(4)}_{\mathrm{R}} t + \gamma^{(4)}_{\mathrm{R}} t^2 + \omega^{(4)}_{\mathrm{R}} (s-u)^2\\ &+ \Delta G^{\mathrm{R}}_{\pi\eta}(s,t;u)\,.\end{aligned} \tag{148}$$

The polynomial part with the coefficients (in what follows, M_S and M_{S_1} are the octet and singlet scalar mass respectively, $c_d, c_m, \tilde{c}_d, \tilde{c}_m$and $\tilde{d}_m$ are the couplings defined in [47])

$$\begin{aligned}\alpha^{(4)}_{\mathrm{R}} = {}& 8M_\pi^2 M_\eta^2\left(-\frac{c_d^2}{3M_S^2}+\frac{2\tilde{c}_d^2}{M_{S_1}^2}\right)\\ &+16M_\pi^2\,\overset{o}{M}{}_\eta^2\left(\frac{c_d c_m}{3M_S^2}-\frac{\tilde{c}_d\tilde{c}_m}{M_{S_1}^2}\right)\\ &+8M_\eta^2\,\overset{o}{M}{}_\pi^2\left(\frac{c_d c_m}{3M_S^2}-\frac{2\tilde{c}_d\tilde{c}_m}{M_{S_1}^2}\right)\\ &-16\frac{\tilde{d}_m^2}{M_{\eta_1}^2}\,\overset{o}{M}{}_\pi^2\left(\overset{o}{M}{}_\pi^2-\overset{o}{M}{}_\eta^2\right)\\ &+20\,\overset{o}{M}{}_\pi^2\overset{o}{M}{}_\eta^2\left(\frac{\tilde{c}_m^2}{M_{S_1}^2}-\frac{c_m^2}{3M_S^2}\right)-M_\pi^2\,\overset{o}{M}{}_\pi^2\,\frac{8c_d c_m}{3M_S^2}\\ &+4\,\overset{o}{M}{}_\pi^4\left(\frac{7c_m^2}{M_S^2}+\frac{\tilde{c}_m^2}{M_{S_1}^2}\right)\,,\end{aligned} \tag{149}$$

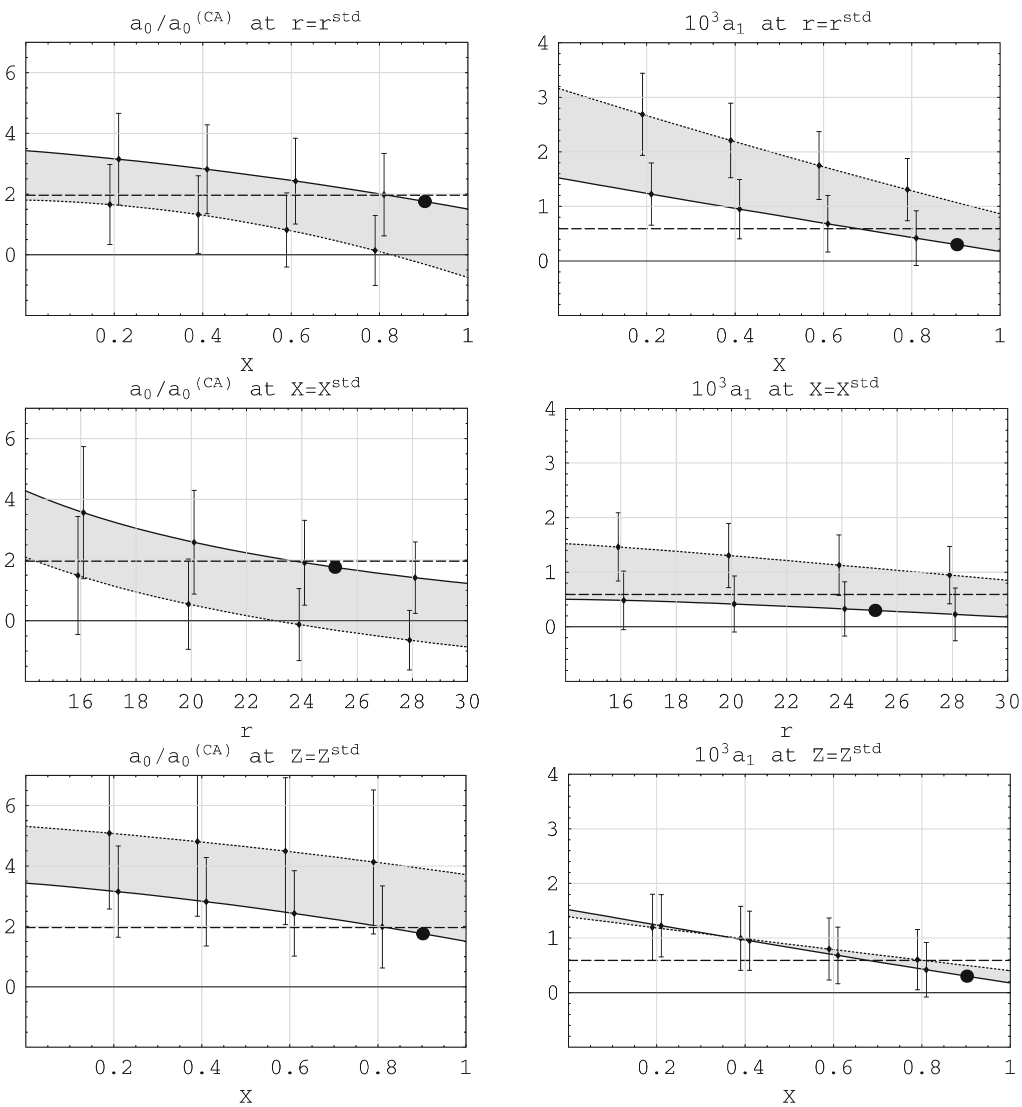

Fig. 5. Dependence of the scattering lengths a_0 and a_1 related to the polynomial parameters α and β. The figures are in one-to-one correspondence to those in Fig. 3

$$\beta_{\mathrm{R}}^{(4)} = -\frac{8\tilde{c}_d\tilde{c}_d}{M_{S_1}^2}\Sigma_{\eta\pi} + \frac{4}{3}\frac{c_d^2}{M_S^2}\Sigma_{\eta\pi} + 8\left(\frac{\tilde{c}_d\tilde{c}_m}{M_{S_1}^2} - \frac{c_d c_m}{3M_S^2}\right)\left(\overset{o}{M}{}_\pi^2 + \overset{o}{M}{}_\eta^2\right), \tag{150}$$

$$\gamma_{\mathrm{R}}^{(4)} = \frac{4\tilde{c}_d^2}{M_{S_1}^2} - \frac{c_d^2}{3M_S^2}, \tag{151}$$

$$\omega_{\mathrm{R}}^{(4)} = \frac{c_d^2}{3M_S^2} \tag{152}$$

gathers the complete $O(p^4)$ resonance contribution (here we can recognize the resonance saturation of the LECs in (90)–(93)). This part of the amplitude is already included in our resummed version of χPT, either explicitly through the LECs $L_1 \ldots L_3$ or implicitly using the reparametrization in terms of the masses, decay constants and parameters r, X and Z. On the other hand, $\Delta G_{\pi\eta,\mathrm{R}}(s,t;u)$ can be formally understood as an infinite sum of the higher order corrections in the (purely) derivative expansion, summed up to

$$\begin{aligned}\Delta G_{\pi\eta,\mathrm{R}}(s,t;u) = {} & \frac{4t}{M_{S_1}^2(M_{S_1}^2 - t)} \\ & \times \left(\tilde{c}_d\left(t - 2M_\pi^2\right) + 2\tilde{c}_m \overset{o}{M}{}_\pi^2\right) \\ & \times \left(\tilde{c}_d\left(t - 2M_\eta^2\right) + 2\tilde{c}_m \overset{o}{M}{}_\eta^2\right) \\ & + \frac{2}{3}\frac{s}{M_S^2(M_S^2 - s)} \\ & \times \left(c_d\left(s - M_\pi^2 - M_\eta^2\right) + 2c_m \overset{o}{M}{}_\pi^2\right)^2 \\ & + \frac{2}{3}\frac{u}{M_S^2(M_S^2 - u)} \\ & \times \left(c_d\left(u - M_\pi^2 - M_\eta^2\right) + 2c_m \overset{o}{M}{}_\pi^2\right)^2 \\ & - \frac{2}{3}\frac{t}{M_S^2(M_S^2 - t)}\end{aligned}$$

$$
\times\left(c_d\left(t-2M_\pi^2\right)+2c_m\,\overset{o}{M}{}_\pi^2\right)
\times\left(c_d(t-2M_\eta^2)+2c_m\left(2\,\overset{o}{M}{}_\eta^2-\overset{o}{M}{}_\pi^2\right)\right)
+16\frac{\tilde{d}_m^2M_\eta^2}{M_{\eta_1}^2(M_{\eta_1}^2-M_\eta^2)}\,\overset{o}{M}{}_\pi^2\left(\overset{o}{M}{}_\eta^2-\overset{o}{M}{}_\pi^2\right). \qquad (153)
$$

Of course, this does not exhaust all possible higher order corrections (note e.g. that the resonance Lagrangian we use contains only the leading order interaction terms with one resonance field and chiral building blocks of the order $O(p^2)$); nevertheless we can use it at least as a rough estimate of the effect of higher orders of the derivative expansion. This is in some sense a procedure opposite to the usual resonance saturation; instead of LECs we "saturate" the remainders by means of sewing together the resummed chiral expansion $G_{\pi\eta}^{\chi\mathrm{PT}}$ (without remainders) with resonance chiral theory, writing the full RχT amplitude as

$$
G_{\pi\eta}^{\mathrm{R}\chi\mathrm{T}}(s,t;u)=G_{\pi\eta}^{\chi\mathrm{PT}}(s,t;u)+\Delta G_{\pi\eta}^{\mathrm{R}}(s,t;u) \qquad (154)
$$

and identifying $G_{\pi\eta}^{\mathrm{R}\chi\mathrm{T}}$ with the full χPT amplitude, the remainder being $\Delta G_{\pi\eta}^{\mathrm{R}}$. Under this assumption, we can derive the following higher order contributions to the direct remainders from $\Delta G_{\pi\eta}^{\mathrm{R}}$

$$
\Delta\alpha_{\mathrm{R}}=\frac{1}{3}F_\pi^2M_\pi^2\delta_\alpha^{\mathrm{R}}=\frac{16}{3}c_m^2\,\overset{o}{M}{}_\pi^4\,\frac{\Sigma_{\eta\pi}}{M_S^2\,(M_S^2-\Sigma_{\eta\pi})}
+16\frac{\tilde{d}_m^2M_\eta^2}{M_{\eta_1}^2\,(M_{\eta_1}^2-M_\eta^2)}\,\overset{o}{M}{}_\pi^2\left(\overset{o}{M}{}_\eta^2-\overset{o}{M}{}_\pi^2\right), \qquad (155)
$$

$$
\begin{aligned}
\Delta\beta_{\mathrm{R}}=\beta\delta_\beta^{\mathrm{R}}=&\frac{16}{M_{S_1}^4}\left(\tilde{c}_dM_\pi^2-\tilde{c}_m\,\overset{o}{M}{}_\pi^2\right)\left(\tilde{c}_dM_\eta^2-\tilde{c}_m\,\overset{o}{M}{}_\eta^2\right)\\
&+\frac{8}{3}\frac{1}{M_S^4}\left(c_dM_\pi^2-c_m\,\overset{o}{M}{}_\pi^2\right)\\
&\times\left(c_m(2\,\overset{o}{M}{}_\eta^2-\overset{o}{M}{}_\pi^2)-c_dM_\eta^2\right)\\
&-\frac{8}{3}\frac{c_m^2\,\overset{o}{M}{}_\pi^4}{M_S^2(M_S^2-\Sigma_{\eta\pi})}-\frac{8}{3}\frac{c_dc_m\,\overset{o}{M}{}_\pi^2\,\Sigma_{\eta\pi}}{M_S^2\,(M_S^2-\Sigma_{\eta\pi})}\\
&-\frac{8}{3}\frac{c_m^2\,\overset{o}{M}{}_\pi^4\,\Sigma_{\eta\pi}}{M_S^2(M_S^2-\Sigma_{\eta\pi})^2}
\end{aligned} \qquad (156)
$$

and similarly for $\delta_\gamma^{\mathrm{R}}$ and $\delta_\omega^{\mathrm{R}}$ (see Appendix J). Note that the dependence on X and Z is exclusively through the ratio $Y=X/Z$ here.

One may notice that there are two distinct features of this procedure as compared to the usual LEC saturation. First, there is no need to fix a saturation scale, which is the result of "saturating" the renormalization scale independent remainder instead of the scale dependent LECs. And second, as the resonance contributions are resummed to all chiral orders, the resonance poles are explicitly present in our result, as can be seen in (153), (155) and (156) as well as the formulae for $\delta_\gamma^{\mathrm{R}}$ and $\delta_\omega^{\mathrm{R}}$ in Appendix J.

For rough numerical estimates we use M_S, M_{S_1}, M_{η_1} and the couplings c_d, c_m, $\tilde{c}_d$, $\tilde{c}_m$ and $\tilde{d}_m$ from [47]. This gives at the standard reference point $X=X^{\mathrm{std}}$, $Z=Z^{\mathrm{std}}$ and $r=r^{\mathrm{std}}$:

$$
\delta_\alpha^{\mathrm{R}}=1.00\,, \qquad (157)
$$

$$
\beta\delta_\beta^{\mathrm{R}}=-0.15\times10^{-3}M_\eta^2\,, \qquad (158)
$$

which represents roughly 55% and 20% correction to the values in the first row of the Table 4, respectively. The dependence of $\delta_\alpha^{\mathrm{R}}$ and $\delta_\beta^{\mathrm{R}}$ on $Y=X/Z$ and r is depicted in Fig. 6. The effect of the resonance remainder estimate on the parameters α and β in a wider range of the X, r and Z is illustrated in the first column of Fig. 8; analogous plots for γ and ω are in Fig. 7. As can be seen, these results suggests the conclusion that the derivative part of the expansion could in some cases produce higher order remainders with a much larger value than 10%.

6.6 Generalized χPT

In the previous subsection we have tried to estimate the contributions to the remainders generated by the derivative expansion. The resulting expressions (155) and (156) could, however, gather only terms of at most the second order[26] in the quark mass expansion due to the lowest order resonance Lagrangian used. Also, the indirect remainders have not been included in this way as there is no contribution to them in this simplest approach. For the appraisal of the importance of the missing terms we therefore need additional information. One possibility might be to use a resonance Lagrangian with additional terms of higher chiral order suited for saturation of the $O(p^6)$ LECs [48] and/or to go to the next-to-next-to leading order in the chiral expansion; this is, however, beyond the scope of our paper.

Instead we try to get some flavor of the size of the effect by means of a comparison of our previous results with generalized χPT, which was originally designed to handle the badly convergent quark mass expansion in the case $X\ll1$ and therefore also includes terms which correspond to higher orders in the standard chiral power counting.

In Sect. 5.4, we have already rewritten the generalized expansion of the parameters α and β (as well as that of the masses and decay constants in Appendices B and I) in the "resummed" form (106)–(109) by means of splitting the "standard" remainders into the "nonstandard" extra terms $\delta^{\mathrm{loops}}(\mu)$ and $\delta^{\mathrm{CT}}(\mu)$ originating in GχPT and the unknown part $\delta^{\mathrm{G}\chi\mathrm{PT}}$. Therefore, neglecting the latter, the sum $\delta^{\mathrm{loops}}(\mu)+\delta^{\mathrm{CT}}(\mu)$ could in a sense be interpreted as a rough estimate of the contribution to the standard remainders stemming from the higher orders of some quark mass expansion.

While the $\delta^{\mathrm{loops}}(\mu)$ are known, the $\delta^{\mathrm{CT}}(\mu)$ depend on the unknown LECs of the GχPT Lagrangian (cf. Appendix F). We therefore set $\delta^{\mathrm{CT}}(\mu)=0$ at the fixed scale

[26] Note that the physical masses in (155) and (156) originate in the derivative expansion.

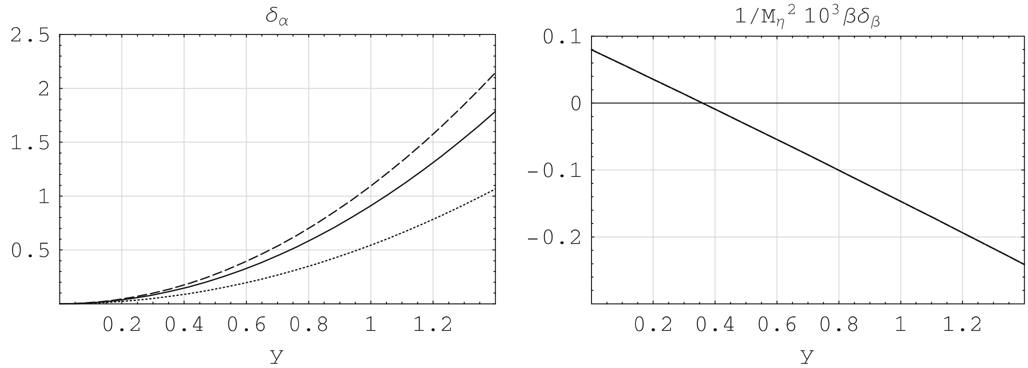

Fig. 6. Dependence of the resonance estimates of the direct remainders on $Y = X/Z$ for $r = 15$ (*dots*), r_2 (*solid*) and 30 (*dashed*). Note that for $M_{S_1} = M_S$ and $\tilde{c}_{m,d} = c_{m,d}/\sqrt{3}$, the remainder $\delta_\beta^{\mathrm{R}}$ is exactly independent on r

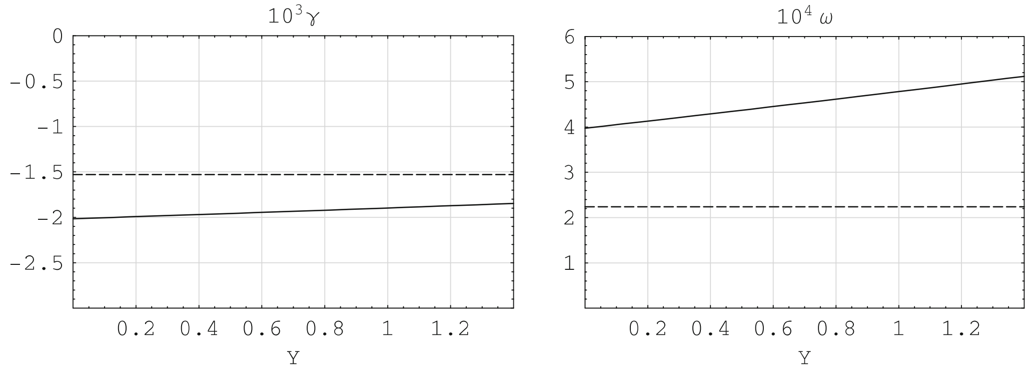

Fig. 7. Polynomial parameters γ and ω depending on Y. *Horizontal dashed line*: standard $O(p^4)$ and central RχPT value. The result with resonance remainder estimates is shown by the *solid line*

μ, and by varying this scale in $\delta^{\mathrm{loops}}(\mu)$ from $\mu = M_\eta$ to $\mu = M_\rho$ we can get some information on the contribution of the unknown LECs (note that $\delta^{\mathrm{loops}}(\mu) + \delta^{\mathrm{CT}}(\mu)$ is renormalization scale independent). We apply this procedure both to the direct and indirect remainders.

The usual way of handling the generalized χPT expansion is to neglect the unknown remainders $\delta^{\mathrm{G}\chi\mathrm{PT}}$. We can repeat the considerations from the previous subsection and partially appreciate them using the resonance estimate. In order to avoid double counting, we have to further modify the resonance contribution to the remainder (153) subtracting terms of the order $O(p^4)$ within the generalized power counting in the same way as was done in the previous subsection (c.f. (148))

$$\begin{aligned}\Delta G^{\mathrm{G}\chi\mathrm{PT}}_{\pi\eta,\mathrm{R}}(s,t;u) &= \Delta G_{\pi\eta,\mathrm{R}}(s,t;u) - \frac{16t}{M_{S_1}^4}\tilde{c}_m^2\,\mathring{M}_\pi^2\mathring{M}_\eta^2 \\ &\quad - \frac{16}{M_S^4}c_m^2\,\mathring{M}_\pi^2\left(\mathring{M}_\pi^2\,(M_\eta^2+M_\pi^2) - \mathring{M}_\eta^2\,t\right) \\ &\quad - \frac{16}{M_{\eta_1}^4}\tilde{d}_m^2\,\mathring{M}_\pi^2\left(\mathring{M}_\eta^2 - \mathring{M}_\pi^2\right)M_\eta^2\,. \quad (159)\end{aligned}$$

This combined GχPT and resonance estimate of the remainders is illustrated in Fig. 8; the right column shows the result in the case of the polynomial parameters α and β. The effect of the unknown GχPT LECs is estimated by their scale dependence. The lines closer to the central $R\chi$PT results with neglected remainders are the ones at the scale $\mu = M_\rho$, i.e. the constants are set to zero at the usually chosen scale. The filled grey areas then show the change when the LECs are set to the difference when moving from the scale $\mu = M_\rho$ to M_η. Admittedly, this assigns quite arbitrary numbers to the LECs, so the uncertainty should be viewed as a rough estimate which can go both ways. The result can be interpreted as being quite consistent with the 10% estimate of the remainders, though clearly exceeding it for some range of the free parameters X, Z and r.

As for the parameters γ and ω, because their contribution in the polynomial expansion is quadratic in the Mandelstam variables, the GχPT estimate does not contribute here.

7 Summary and conclusions

In this paper we have studied the properties of various variants of the chiral expansion, namely the recently introduced resummed χPT as compared to the standard and partly generalized versions, on the concrete example of $\pi\eta$ scattering. Our calculations paid special attention to the possible reparametrization in terms of the physical observables. We have tried to illustrate several issues in detail, specifically the following.

- We considered the necessity of carefully choosing a class of "good" observables for which the condition of global convergence is believed to be satisfied in the sense that the $O(p^6)$ and higher remainders are small and under control.
- Next, we have illustrated the necessity to carefully define the bare expansion of "good" observables. Here we have concentrated on the requirements dictated by the

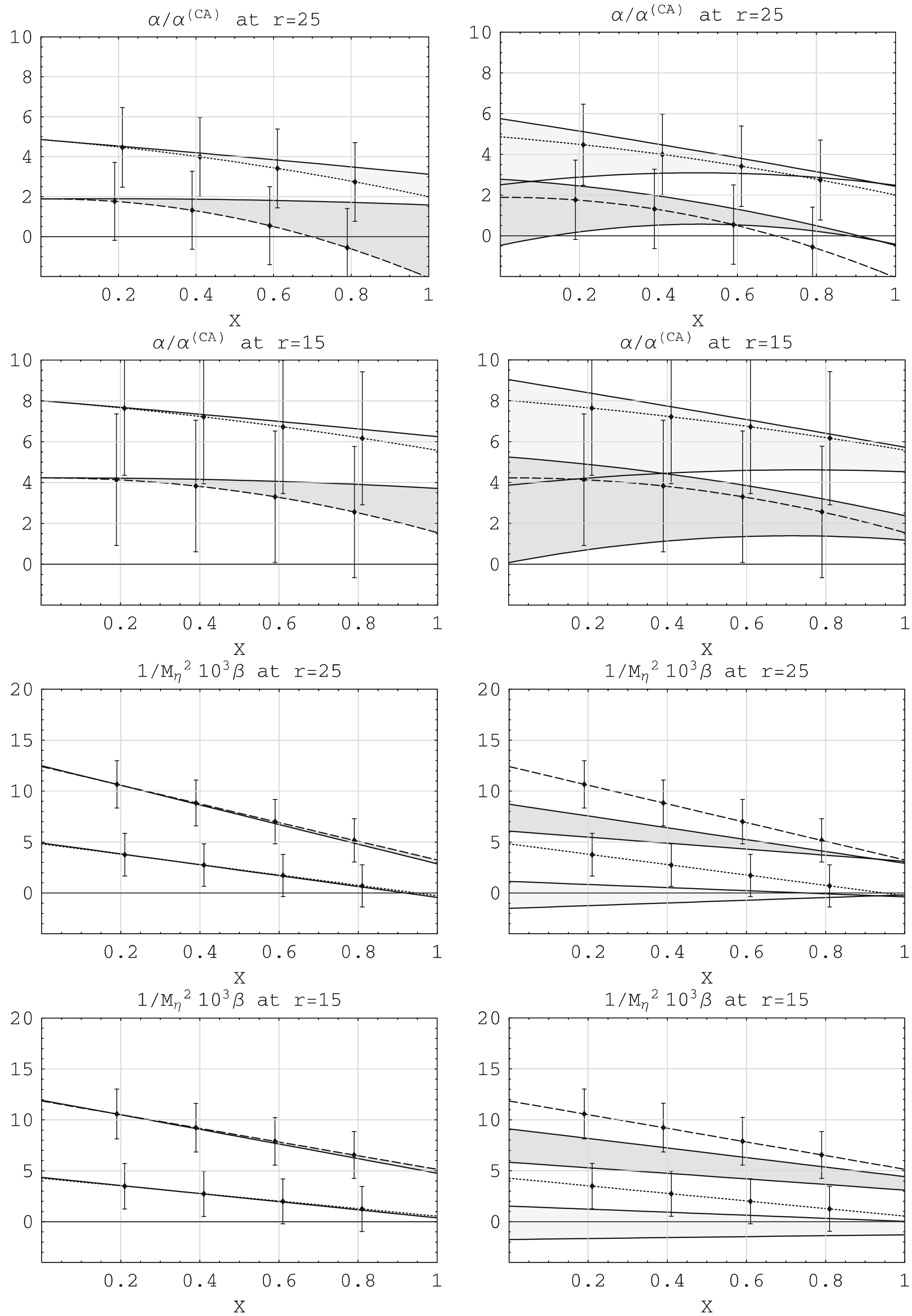

Fig. 8. Polynomial parameters α and β depending on X and Z for traditional and low values of r. The *dotted line* shows the central value for $Z = 0.9$, the *dashed one* is the same for $Z = 0.5$. The *error bars* correspond to the 10% estimates of the remainders. *Left column*: resonance estimate, *filled areas* highlight the $O(p^6)$ and higher corrections to the amplitude generated by resonances (*lighter* for $Z = 0.9$, *darker* for $Z = 0.5$). *Right column*: results with combined resonance and GχPT estimate of remainders. *Filled areas* show the scale dependence ($\mu \sim M_\eta - M_\rho$), *lighter* ones are for $Z = 0.9$ and *darker* ones for $Z = 0.5$

exact renormalization scale independence as well as the exact perturbative unitarity. As we have shown, both these requirements can be met by means of sewing together the strict chiral expansion in terms of the LECs with the dispersive representation for the corresponding Green function. Nevertheless, the resulting bare expansion is not yet defined uniquely; one has to fix the way how to treat the chiral logs and also the $O(p^2)$ amplitudes entering the dispersive integrals. Though the difference is formally of the same order as the remainder itself, we have found that it might be numerically significant in some region of the free parameters.

– We have investigated the properties of the standard chiral expansion, based on the potentially "dangerous" reparametrization of the bare expansion implicitly assuming $X, Z \sim 1$ and $r \sim r_2$. In this case we have established a strong sensitivity of the observables for $\pi\eta$ scattering on the $O(p^4)$ LECs; this plagues the standard prediction with a large uncertainty. In the case of L_4 and L_6 this also means a strong sensitivity to the vacuum fluctuations of the $\overline{s}s$ pairs and therefore to the deviation from the standard scenario with $X, Z \sim 1$. The unusually large absolute values of the NLO corrections as well as large variations achieved for most of the observables (including the "good" ones) when moving from the $O(p^4)$ fit of the LECs L_i [44] to the $O(p^6)$ based fit [45, 46] might be interpreted as a signal of the importance of the NNLO corrections within the standard chiral expansion. This seems to be also supported by the sensitivity of the NLO contributions to their form when expressed in terms of the physical masses and decay constants (i.e. how the $O(p^2)$ relations like e.g. the Gell-Mann–Okubo formula are used).

- We have considered the properties of the "safe" reparametrization and resummation of the vacuum fluctuations. We have confirmed that, for the "good" observables, the resummed and standard values coincide near the standard reference point $[X^{\mathrm{std}}, Z^{\mathrm{std}}, r^{\mathrm{std}}]$. Under our working hypothesis, which assumes the "good" observables to be accompanied with small and controllable remainders, this can be interpreted as consistence of the standard $O(p^4)$ chiral expansion of "good" observables in the sense that the potentially large higher order remainders are in fact small. On the other hand, in most cases of the "dangerous" observables the standard and resummed values do not meet at $[X^{\mathrm{std}}, Z^{\mathrm{std}}, r^{\mathrm{std}}]$ (typically for the ones that start at $O(p^4)$). Though this might indicate that the standard expansion is convergent less satisfactorily in this case and the higher order remainders might be important here, the difference between the standard and resummed values lies within the estimated uncertainty of the resummed prediction. Away from the standard reference point, however, we have established that the central values diverge substantially from those of the standard approach even for the "good" observables. This is a signal that, unless $X, Z \sim 1$, the higher order remainders of the standard chiral expansion might be huge in comparison with LO + NLO value. Though this feature does not exclude the possibility that in this case the standard remainders might be saturated by the NNLO corrections, it could nevertheless be interpreted as an indication of the instability of the standard chiral expansion.
- We have discussed the role of the remainders within the resummed approach. We have found the strong sensitivity of the observables connected to $\pi\eta$ scattering to the higher order remainders. This might reduce the predictive power of this approach, unless additional information on the actual size of the remainders is available. We have tried to make an independent estimate of the remainders using the simplest version of the resonance chiral Lagrangian as well as making a comparison with GχPT. Both these estimates seem to be in accord with the rough expectation $\delta \sim 10\%$ for the remainders only in some range of the parameters. For some observables and some corners of the parameter space, they can be substantially larger. Of course, the convergence properties of the bare expansion deserves further investigation by means of going to the NNLO, which is, however, beyond the scope of our article.

Let us add some final remarks concerning the interpretation of the above results from a practical point of view. The resummed version of the χPT expansion not only seems to be a suitable framework for taking the effect of large $s\overline{s}$ pair vacuum fluctuations into account, but by keeping the remainders as explicit parameters it effectively includes all orders of the chiral expansion and thus it opens a space for incorporating further improvements of the predictions using additional information from various sources. As our analysis shows, $\pi\eta$ scattering allows one to test the plausibility of the standard assumption $X, Z \sim 1$, $r \sim 25$ due to the sensitivity of the corresponding observables to the deviation of X, Z and r from these values. Provided the experimental data were available, this could be done purely in the resummed framework using statistical methods similar to the ones used in the cases of $\pi\pi$ and πK scattering [8, 9].

On the other hand, to resolve a direct disagreement between the standard and resummed predictions is more delicate. At first sight, even though the SχPT corrections at the NNLO are still not available, the possible experimental data which were in conflict with the standard $O(p^4)$ prediction but still compatible with that of resumed χPT might indicate problems with the standard chiral expansion based on the assumption $X, Z \sim 1$, $r \sim 25$. This might show up as unusually large $O(p^6)$ corrections or as $O(p^6)$ corrections too small to saturate the standard remainders. However, as we have illustrated in Sect. 6.1, the central values of the standard $O(p^4)$ predictions are plagued with large uncertainties even for the "good" observables. This feature together with the lack of information concerning the size of the standard $O(p^6)$ corrections would most likely prevent us from making a decisive conclusion concerning the possible deviations of the resummed χPT from the standard chiral expansion. In the light of our results, this is expected – bad convergence of the standard chiral expansion does not necessarily manifest itself as a direct conflict with the experimental data at NLO, but rather in large uncontrollable uncertainties attached to its predictions.

Because of the current lack of low energy $\pi\eta$ scattering data, the comparison with experiment can only be done indirectly. As we have mentioned in the Introduction, the promising process here is the rare decay $\eta \to \pi^0\pi^0\gamma\gamma$, where the off-shell $\pi\eta\pi\eta^*$ vertex enters the nonresonant part of the amplitude. As preliminary studies [35, 36] using GχPT show, the effect of the $s\overline{s}$ pair vacuum fluctuations parametrized by X, Z away from their standard values might give large deviations from the prediction of SχPT [49–51], resulting in the increase of the η-tail of the diphoton spectrum, which can be in principle observed. Based on the above results, the more careful analysis using a resummed version of χPT expansion is expected to yield qualitatively the same effect [37].

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Appendix A: Standard chiral expansion of parameters X, Z and r

Here we summarize the formulae leading to the standard values of X, Z, Y and r used in Sect. 6.3. Using the standard reparametrization rules explained in Sect. 5.1, we get up to the NLO order in terms of the $O(p^4)$ LECs

$$\begin{aligned}X^{\text{std}} &= 1 - \frac{M_\pi^2}{2F_\pi^2}\bigg(32\left(L_6^r(r_2+2)+L_8^r\right)+3J_{\pi\pi}^r(0)+(r_2+1)\\ &\quad\times J_{KK}^r(0)+\frac{1}{9}(2r_2+1)J_{\eta\eta}^r(0)+\frac{11r_2+37}{144\pi^2}\bigg),\\ Z^{\text{std}} &= 1-\frac{M_\pi^2}{2F_\pi^2}\bigg(16\left(L_4^r(r_2+2)+L_5^r\right)+(r_2+1)J_{KK}^r(0)\\ &\quad+4J_{\pi\pi}^r(0)+\frac{r_2+5}{16\pi^2}\bigg),\\ r^{\text{std}} &= r_2-\frac{M_\pi^2(r_2+1)}{2F_\pi^2}\bigg(8\left(2L_8^r-L_5\right)(r_2-1)\\ &\quad-\frac{1}{3}(2r_2+1)J_{\eta\eta}^r(0)+J_{\pi\pi}^r(0)-\frac{(r_2-1)}{24\pi^2}\bigg).\end{aligned} \tag{A.1}$$

For r^{std} we can also use an alternative expression based on the chiral expansion of r_2^*:

$$\begin{aligned}r^{*\text{std}} &= r_2^*-\frac{M_\pi^2(r_2+1)}{2F_\pi^2}\bigg(16L_8^r(r_2-1)+\frac{1}{6}(2r_2+1)J_{\eta\eta}^r(0)\\ &\quad+\frac{1}{2}(r_2+1)J_{KK}^r(0)-\frac{3}{2}J_{\pi\pi}^r(0)+\frac{5(r_2-1)}{96\pi^2}\bigg).\end{aligned} \tag{A.2}$$

Δ_{GMO} has the following standard chiral expansion:

$$\begin{aligned}\Delta_{\text{GMO}}^{\text{std}} &= \frac{M_\pi^2(r_2-1)}{2F_\pi^2}\bigg(32\left(2L_7^r+L_8^r\right)(r_2-1)\\ &\quad+\frac{1}{3}(2r_2+1)J_{\eta\eta}^r(0)+(r_2+1)J_{KK}^r(0)-3J_{\pi\pi}^r(0)\\ &\quad+\frac{5(r_2-1)}{48\pi^2}\bigg).\end{aligned} \tag{A.3}$$

Appendix B: Chiral expansion of the η decay constant

For the bare expansion of the "good" observables F_P^2 we rewrite the standard formulae in the form

$$\begin{aligned}F_\pi^2 &= F_0^2\bigg[1+\frac{B_0\widehat{m}}{F_0^2}\bigg(16L_4^r(\mu)(r+2)+16L_5^r(\mu)\\ &\quad+(r+1)J_{KK}^r(0)+4J_{\pi\pi}^r(0)+\frac{1}{16\pi^2}(r+5)\bigg)\bigg]\\ &\quad+F_\pi^2\delta_{F_\pi},\end{aligned} \tag{B.1}$$

$$\begin{aligned}F_K^2 &= F_0^2\bigg[1+\frac{B_0\widehat{m}}{F_0^2}\bigg(16L_4^r(\mu)(r+2)+8L_5^r(\mu)(r+1)\\ &\quad+\frac{3}{2}(r+1)\left(J_{KK}^r(0)+\frac{1}{16\pi^2}\right)+\frac{3}{2}\left(J_{\pi\pi}^r(0)+\frac{1}{16\pi^2}\right)\\ &\quad+\frac{1}{2}(2r+1)\left(J_{\eta\eta}^r(0)+\frac{1}{16\pi^2}\right)\bigg)\bigg]+F_K^2\delta_{F_K},\end{aligned} \tag{B.2}$$

$$\begin{aligned}F_\eta^2 &= F_0^2\bigg[1+\frac{B_0\widehat{m}}{F_0^2}\bigg(16L_4^r(\mu)(r+2)+\frac{16}{3}L_5^r(\mu)(2r+1)\\ &\quad+3(r+1)\left(J_{KK}^r(0)+\frac{1}{16\pi^2}\right)\bigg)\bigg]+F_\eta^2\delta_{F_\eta}.\end{aligned} \tag{B.3}$$

Within standard χPT, the $O(p^2)$ parameters B_0, F_0 and r are expressed using inverted expansions of the observables F_P^2 and M_P^2, as explained in Sect. 5.1. This yields the standard formula for F_η^2:

$$\begin{aligned}F_\eta^2 &= F_\pi^2\bigg[1+\frac{M_\pi^2}{F_\pi^2}\bigg(\frac{16}{3}L_5^r(\mu)(r_2-1)+(r_2+1)J_{KK}^r(0)\\ &\quad-2J_{\pi\pi}^r(0)+\frac{1}{16\pi^2}(r_2-1)\bigg)\bigg]+F_\eta^2\delta_{F_\eta}^{\text{st}},\end{aligned} \tag{B.4}$$

with a potentially large remainder $\delta_{F_\eta}^{\text{st}}$. Numerically, with $L_5^r(M_\rho)$ taken from [44] we get

$$F_\eta^2 = 1.625F_\pi^2. \tag{B.5}$$

On the other hand, the "safe" reparametrization in terms of r, X and Z gives

$$\begin{aligned}F_\eta^2 &= F_\pi^2\bigg[1+\frac{2}{3}(r-1)\eta(r)-\frac{1}{3}\frac{M_\pi^2}{F_\pi^2}\left(\frac{X}{Z}\right)\\ &\quad\times\left(J_{\pi\pi}^r(0)-2(r+1)J_{KK}^r(0)+(2r+1)J_{\eta\eta}^r(0)\right)\bigg]\\ &\quad+\frac{1}{3}\left(3F_\eta^2\delta_{F_\eta}+F_\pi^2\delta_{F_\pi}-4F_K^2\delta_{F_K}\right),\end{aligned} \tag{B.6}$$

which is valid as an exact algebraic identity.[27]

[27] This identity can be also rewritten as

$$\begin{aligned}&4F_K^2(1-\delta_{F_K})-F_\pi^2(1-\delta_{F_\pi})-3F_\eta^2(1-\delta_{F_\eta})\\ &=\left(\frac{X}{Z}\right)M_\pi^2\left(J_{\pi\pi}^r(0)-2(r+1)J_{KK}^r(0)+(2r+1)J_{\eta\eta}^r(0)\right).\end{aligned}$$

Within the standard approach, the parameters on the r.h.s. of this identity can be expressed to the order $O(p^4)$ in terms of the physical observables, and it is interpreted as a $O(p^4)$ sum rule:

$$\begin{aligned}4F_K^2-F_\pi^2-3F_\eta^2 &= M_\pi^2\big(J_{\pi\pi}^r(0)-2(r_2+1)J_{KK}^r(0)\\ &\quad+(2r_2+1)J_{\eta\eta}^r(0)\big).\end{aligned}$$

This gives

$$F_\eta^2 = 1.697F_\pi^2.$$

Following the GχPT procedure outlined in Sect. 5.4, after identifying the corresponding LECs in both approaches

$$\frac{B_0\widehat{m}}{F_0^2}L_4^r(\mu) \to \frac{1}{8}\widehat{m}\tilde{\xi}\,,\qquad \frac{B_0\widehat{m}}{F_0^2}L_5^r(\mu) \to \frac{1}{8}\widehat{m}\xi^r\,, \tag{B.7}$$

and defining the remainders using the physical masses inside the chiral logs, we can use the exact formula (B.3) and write the remainder δ_{F_η} within GχPT as

$$F_\eta^2\delta_{F_\eta} = F_\eta^2\delta_{F_\eta}^{\text{loop}}(\mu) + F_0^2\delta_{F_\eta}^{(4)\text{CT}}(\mu) + F_\eta^2\delta_{F_\eta}^{\text{G}\chi\text{PT}}\,.$$

Here

$$F_\eta^2\delta_{F_\eta}^{\text{loop}}(\mu) = 3\widehat{m}^2(r+1)\left(A_0(r+1)+2(r+2)Z_0^S\right)\times\left(J_{KK}^r(0)+\frac{1}{16\pi^2}\right) \tag{B.8}$$

is the extra loop contribution and $\delta_{F_\eta}^{(4)\text{CT}}(\mu)$ the contribution of the counterterms from the $O(p^4)$ GχPT Lagrangian renormalized at the scale μ:

$$\delta_{F_\eta}^{(4)\text{CT}}(\mu) = \frac{2}{3}\widehat{m}^2\left[\frac{1}{2}(2A_1+A_2+4A_3+2B_1-2B_2)\times(1+2r^2)+3(A_4+2B_4)\left(1+\frac{1}{2}r^2\right) - 4C_1^P(r-1)^2+2D^S(r+2)(2r+1)\right]. \tag{B.9}$$

$\delta_{F_\eta}^{\text{G}\chi\text{PT}}$ is a new remainder, which is exactly independent on the renormalization scale.

Analogously, for the GχPT formula for F_π^2 we have, besides the substitution (B.7) to (B.1), to insert

$$F_\pi^2\delta_{F_\pi} = F_\pi^2\delta_{F_\pi}^{\text{loop}}(\mu) + F_0^2\delta_{F_\pi}^{(4)\text{CT}}(\mu) + F_\pi^2\delta_{F_\pi}^{\text{G}\chi\text{PT}}\,, \tag{B.10}$$

where the loop and counterterm contribution are now

$$F_\pi^2\delta_{F_\pi}^{\text{loop}}(\mu) = 8\widehat{m}^2\left(A_0+(r+2)Z_0^S\right))\left(J_{\pi\pi}^r(0)+\frac{1}{16\pi^2}\right) + \widehat{m}^2(r+1)\left(A_0(r+1)+2(r+2)Z_0^S\right)\times\left(J_{KK}^r(0)+\frac{1}{16\pi^2}\right),$$

$$\delta_{F_\pi}^{(4)CT}(\mu) = 2\widehat{m}^2\left[A_1+\frac{1}{2}A_2+2A_3+(A_4+2B_4)\times\left(1+\frac{1}{2}r^2\right)+B_1-B_2+2D^S(r+2)\right]. \tag{B.11}$$

Finally, we have the expression for F_K^2, where the remainder is replaced with

$$F_K^2\delta_{F_K} = F_K^2\delta_{F_K}^{\text{loop}}(\mu) + F_0^2\delta_{F_K}^{(4)\text{CT}}(\mu) + F_K^2\delta_{F_K}^{\text{G}\chi\text{PT}} \tag{B.12}$$

and the loops and counterterms contribute as follows:

$$F_K^2\delta_{F_K}^{\text{loop}}(\mu) = 3\widehat{m}^2(A_0+(r+2)Z_0^S))\left(J_{\pi\pi}^r(0)+\frac{1}{16\pi^2}\right) + \frac{3}{2}\widehat{m}^2(r+1)\left(A_0(r+1)+2(r+2)Z_0^S\right)\times\left(J_{KK}^r(0)+\frac{1}{16\pi^2}\right) + \widehat{m}^2\big(A_0(2r^2+1) + 2(r-1)^2Z_0^P+(r+2)(2r+1)Z_0^S\big)\times\left(J_{\eta\eta}^r(0)+\frac{1}{16\pi^2}\right),$$

$$\delta_{F_K}^{(4)\text{CT}}(\mu) = \widehat{m}^2\left[(A_1+B_1)(r^2+1)+(A_2-2B_2)r + 2(A_4+2B_4)\left(1+\frac{1}{2}r^2\right) + 2D^S(r+2)(r+1)\right]. \tag{B.13}$$

In order to reparametrize the GχPT bare expansion in terms of the masses and decay constants, we can proceed as follows. Because the exact identity (B.6) is valid independently of the version of χPT, we can also use it in the generalized case, provided we rewrite the remainders according to (B.9), (B.10) and (B.12). This step eliminates the LECs ξ and $\tilde{\xi}$. Collecting the chiral logs we have

$$F_\eta^2 = F_\pi^2\left[1+\frac{2}{3}(r-1)\eta(r)-\frac{1}{3F_\pi^2}\left(\tilde{M}_\pi^2\left(J_{\pi\pi}^r(0)+\frac{1}{16\pi^2}\right) - 4\tilde{M}_K^2\left(J_{KK}^r(0)+\frac{1}{16\pi^2}\right) + 3\tilde{M}_\eta^2\left(J_{\eta\eta}^r(0)+\frac{1}{16\pi^2}\right)\right)\right] + F_\eta^2\Delta_{F_\eta}^{\text{G}\chi\text{PT}}(\mu)\,, \tag{B.14}$$

where the $O(p^2)$ masses are given by (112) and

$$F_\eta^2\Delta_{F_\eta}^{\text{G}\chi\text{PT}}(\mu) = \frac{1}{3}F_0^2\left(3\delta_{F_\eta}^{(4)\text{CT}}+\delta_{F_\pi}^{(4)\text{CT}}-4\delta_{F_K}^{(4)\text{CT}}\right) + \frac{1}{3}\Big(3F_\eta^2\delta_{F_\eta}^{\text{G}\chi\text{PT}}+F_\pi^2\delta_{F_\pi}^{\text{G}\chi\text{PT}} - 4F_K^2\delta_{F_K}^{\text{G}\chi\text{PT}}\Big) \tag{B.15}$$

The last step consists of replacing the LECs F_0, A_0, Z_0^S and Z_0^P with the first term of their expansion in terms of the masses and decay constants as described in Sect. 5.4. This corresponds to a further redefinition of the generalized remainders.

Appendix C: Dispersion representation of the $\pi\eta$ amplitude

For the dispersive representation of the amplitude we need the S- and T-channel discontinuities at $O(p^4)$. In the following subsections we give a list of the relevant $O(p^2)$ amplitudes $G^{(2)A_\text{i}\to ij}$ and $G^{(2)ij\to A_\text{f}}$ and the $O(p^4)$ discontinuities disc G_0^{ij} corresponding to the different intermediate states ij.

C.1 *S*-channel discontinuities at $O(p^4)$

– For the $\pi\eta$ intermediate state we have

$$G^{(2)\pi\eta\to\pi\eta} = \frac{1}{3}F_0^2 \overset{o}{M}{}_\pi^2$$
$$\operatorname{disc} G_0^{\pi\eta}(s) = 2\frac{\lambda^{1/2}\left(s, M_\pi^2, M_\eta^2\right)}{s}\left(\frac{1}{32\pi}\frac{1}{3}\overset{o}{M}{}_\pi^2\right)^2 \frac{F_0^4}{F_\pi^2 F_\eta^2}. \tag{C.1}$$

– For the $\overline{K}K$ intermediate state we have

$$G^{(2)\pi\eta\to\overline{K^0}K^0(K^+K^-)} = -\frac{\sqrt{3}}{4}F_0^2 \times\left(s-\frac{1}{3}M_\eta^2-\frac{1}{3}M_\pi^2-\frac{2}{3}M_K^2\right) + \frac{1}{4\sqrt{3}}F_0^2\left(2\overset{o}{M}{}_K^2-\overset{o}{M}{}_\pi^2-\overset{o}{M}{}_\eta^2\right),$$
$$G^{(2)\overline{K^0}K^0(K^+K^-)\to\pi\eta} = -\frac{\sqrt{3}}{4}F_0^2 \times\left(s-\frac{1}{3}M_\eta^2-\frac{1}{3}M_\pi^2-\frac{2}{3}M_K^2\right) + \frac{1}{4\sqrt{3}}F_0^2\left(2\overset{o}{M}{}_K^2-\overset{o}{M}{}_\pi^2-\overset{o}{M}{}_\eta^2\right),$$
$$\operatorname{disc} G_0^{\overline{K^0}K^0(K^+K^-)}(s) = 2\sqrt{1-\frac{4M_K^2}{s}} \times\left(\frac{1}{32\pi}\right)^2\frac{3}{16}\left[\left(s-\frac{1}{3}M_\eta^2-\frac{1}{3}M_\pi^2-\frac{2}{3}M_K^2\right) -\frac{1}{3}\left(2\overset{o}{M}{}_K^2-\overset{o}{M}{}_\pi^2-\overset{o}{M}{}_\eta^2\right)\right]^2\frac{F_0^4}{F_K^4}. \tag{C.2}$$

C.2 *T*-channel discontinuities at $O(p^4)$

– For the $\pi\pi$ intermediate state we have

$$G^{(2)\pi\pi\to\pi\pi,I=0} = F_0^2\left[\left(s-\frac{4}{3}M_\pi^2\right)+\frac{5}{6}\overset{o}{M}{}_\pi^2\right],$$
$$\operatorname{disc} G_0^{\pi\pi,I=0}(s) = 2\sigma(s)\left(\frac{1}{32\pi}\right)^2\frac{1}{3}\overset{o}{M}{}_\pi^2 \times\left[\left(s-\frac{4}{3}M_\pi^2\right)+\frac{5}{6}\overset{o}{M}{}_\pi^2\right]\frac{F_0^4}{F_\pi^4}. \tag{C.3}$$

– For the $\eta\eta$ intermediate state we have

$$G^{(2)\eta\eta\to\eta\eta} = -\frac{1}{3}F_0^2\left(\overset{o}{M}{}_\pi^2-4\overset{o}{M}{}_\eta^2\right),$$
$$\operatorname{disc} G_0^{\eta\eta}(s) = -2\frac{1}{2}\sqrt{1-\frac{4M_\eta^2}{s}}\left(\frac{1}{32\pi}\right)^2 \times\frac{1}{9}\overset{o}{M}{}_\pi^2\left(\overset{o}{M}{}_\pi^2-4\overset{o}{M}{}_\eta^2\right)\frac{F_0^4}{F_\eta^4}. \tag{C.4}$$

– For the $\overline{K}K$ intermediate state[28] we have

$$G^{(2)\pi\pi\to\overline{K^0}K^0(K^+K^-),I=0} = \mp\frac{\sqrt{3}}{4}F_0^2\left[\left(s-\frac{2}{3}M_\pi^2-\frac{2}{3}M_K^2\right)+\frac{2}{3}\left(\overset{o}{M}{}_K^2+\overset{o}{M}{}_\pi^2\right)\right],$$
$$G^{(2)\overline{K^0}K^0(K^+K^-)\to\eta\eta,I=0} = \pm\frac{1}{4}F_0^2\left[\left(3s-2M_K^2-2M_\eta^2\right)+\left(2\overset{o}{M}{}_\eta^2-\frac{2}{3}\overset{o}{M}{}_K^2\right)\right],$$
$$\operatorname{disc} G_0^{\overline{K^0}K^0(K^+K^-),I=0}(s) = \sqrt{1-\frac{4M_K^2}{s}}\left(\frac{1}{32\pi}\right)^2\frac{1}{16}\left[\left(s-\frac{2}{3}M_\pi^2-\frac{2}{3}M_K^2\right) +\frac{2}{3}\left(\overset{o}{M}{}_K^2+\overset{o}{M}{}_\pi^2\right)\right]\left[\left(3s-2M_K^2-2M_\eta^2\right) +\left(2\overset{o}{M}{}_\eta^2-\frac{2}{3}\overset{o}{M}{}_K^2\right)\right]\frac{F_0^4}{F_K^4}. \tag{C.5}$$

Appendix D: The scalar bubble

In this appendix we summarize the formulae for the scalar bubble, defined as

$$J_{PQ}(q^2) = -\mathrm{i}\int\frac{\mathrm{d}^d k}{(2\pi)^d} \times\frac{1}{(k^2-M_P^2+\mathrm{i}0)\left((k-q)^2-M_Q^2+\mathrm{i}0\right)} = -2\lambda_\infty + J_{PQ}^r(q^2). \tag{D.1}$$

Here, as usual

$$\lambda_\infty = \frac{\mu^{d-4}}{16\pi^2}\left(\frac{1}{d-4}-\frac{1}{2}(\ln 4\pi+\Gamma'(1)+1)\right) \tag{D.2}$$

and $J_{PQ}^r(s) = J_{PQ}^r(0)+\overline{J}_{PQ}(s)$, where

$$J_{PQ}^r(0) = -\frac{1}{16\pi^2}\frac{M_P^2\ln(M_P^2/\mu^2)-M_Q^2\ln(M_Q^2/\mu^2)}{M_P^2-M_Q^2} \tag{D.3}$$

and $\overline{J}_{PQ}(s)$, sometimes called the Chew–Mandelstam function, can be expressed by means of the once subtracted dispersion relation as

$$\overline{J}_{PQ}(s) = \frac{s}{16\pi^2}\int_{(M_P+M_Q)^2}^\infty\frac{\mathrm{d}x}{x}\frac{\lambda^{1/2}(x,M_P^2,M_Q^2)}{x}\frac{1}{x-s}. \tag{D.4}$$

[28] Let us note that

$$G^{I=0}(s,t;u) = -\frac{1}{\sqrt{3}}\delta^{ab}G^{ab}(s,t;u) = -\sqrt{3}G(s,t;u).$$

The explicit form of $\overline{J}_{PQ}(s)$ reads

$$\overline{J}_{PQ}(s)=\frac{1}{32\pi^2}\left(2+\frac{\Delta_{PQ}}{s}\ln\frac{M_Q^2}{M_P^2}-\frac{\Sigma_{PQ}}{\Delta_{PQ}}\ln\frac{M_Q^2}{M_P^2}+2\frac{\left(s-(M_P-M_Q)^2\right)}{s}\sigma_{PQ}(s)\ln\frac{\sigma_{PQ}(s)-1}{\sigma_{PQ}(s)+1}\right), \quad \text{(D.5)}$$

where

$$\Delta_{PQ}=M_P^2-M_Q^2,$$
$$\Sigma_{PQ}=M_P^2+M_Q^2,$$
$$\sigma_{PQ}(t)=\sqrt{\frac{s-(M_P+M_Q)^2}{s-(M_P-M_Q)^2}}=\sqrt{1-\frac{4M_PM_Q}{s-(M_P-M_Q)^2}}. \quad \text{(D.6)}$$

In the limit $M_P\to M_Q$ we get

$$J_{PP}^r(0)=-\frac{1}{16\pi^2}\left(\ln\frac{M_P^2}{\mu^2}+1\right)$$
$$\overline{J}_{PP}(s)=\frac{1}{16\pi^2}\left(2+\sigma_{PP}(s)\ln\frac{\sigma_{PP}(s)-1}{\sigma_{PP}(s)+1}\right). \quad \text{(D.7)}$$

Appendix E: $O(p^4)$ constants L_4–L_8 in terms of masses and decay constants

In this appendix we summarize the formulae used in the text for the reparametrization of bare expansions of "good" observables. We use the abbreviated notation (102)–(105). From the bare expansion of "good" variables F_π^2 and F_K^2 we obtain

$$4\overset{o}{M}{}_\pi^2\,L_4^r(\mu)=\frac{1}{2}(1-Z-\eta(r))\frac{F_\pi^2}{r+2}-\frac{M_\pi^2}{4(r+2)(r-1)}\frac{X}{Z}\times\left[(4r+1)J_{\pi\pi}^r(0)+(r-2)(r+1)J_{KK}^r(0)-(2r+1)J_{\eta\eta}^r(0)+\frac{(r+2)(r-1)}{16\pi^2}\right]+\frac{2F_K^2\delta_{F_K}-(r+1)F_\pi^2\delta_{F_\pi}}{2(r+2)(r-1)}, \quad \text{(E.1)}$$

$$4\overset{o}{M}{}_\pi^2\,L_5^r(\mu)=\frac{1}{2}F_\pi^2\eta(r)+\frac{M_\pi^2}{4(r-1)}\frac{X}{Z}\left[5J_{\pi\pi}^r(0)-(r+1)\times J_{KK}^r(0)-(2r+1)J_{\eta\eta}^r(0)-\frac{3(r-1)}{16\pi^2}\right]-(r+1)-\frac{F_K^2\delta_{F_K}-F_\pi^2\delta_{F_\pi}}{(r-1)}. \quad \text{(E.2)}$$

In the same way, from the expansion of $F_P^2M_P^2$ we get

$$4\overset{o}{M}{}_\pi^4\,L_6^r(\mu)=\frac{1}{4}\frac{F_\pi^2M_\pi^2}{r+2}(1-X-\varepsilon(r))-\frac{M_\pi^4}{72(r-1)(r+2)}\times\left(\frac{X}{Z}\right)^2\left[27rJ_{\pi\pi}^r(0)+9(r+1)(r-2)J_{KK}^r(0)+(2r+1)(r-4)J_{\eta\eta}^r(0)+\frac{11(r-1)(r+2)}{16\pi^2}\right]-\frac{F_\pi^2M_\pi^2\delta_{F_\pi M_\pi}[(r+1)^2]-4F_K^2M_K^2\delta_{F_KM_K}}{4(r^2-1)(r+2)}, \quad \text{(E.3)}$$

$$4\overset{o}{M}{}_\pi^4\,L_7^r(\mu)=-\frac{1}{8}F_\pi^2M_\pi^2\left(\varepsilon(r)-\frac{\Delta_{\mathrm{GMO}}}{(r-1)^2}\right)-\left\{3(1+r)F_\eta^2M_\eta^2\delta_{F_\eta M_\eta}+(2r^2+r-1)F_\pi^2M_\pi^2\delta_{F_\pi M_\pi}-8rF_K^2M_K^2\delta_{F_KM_K}\right\}\frac{1}{8(r-1)^2(r+1)}, \quad \text{(E.4)}$$

$$4\overset{o}{M}{}_\pi^4\,L_8^r(\mu)=\frac{1}{4}F_\pi^2M_\pi^2\varepsilon(r)+\frac{M_\pi^4}{24(r-1)}\left(\frac{X}{Z}\right)^2\times\left[9J_{\pi\pi}^r(0)-3(r+1)J_{KK}^r(0)-(2r+1)\times J_{\eta\eta}^r(0)-\frac{5(r-1)}{16\pi^2}\right]-\frac{2F_K^2M_K^2\delta_{F_KM_K}-(r+1)F_\pi^2M_\pi^2\delta_{F_\pi M_\pi}}{2(r^2-1)}. \quad \text{(E.5)}$$

Appendix F: Lagrangian of GχPT to $O(p^4)$

Here we give the traditional form of the GχPT Lagrangian. In the following we use the notation:

$$\chi=\mathcal{M}+s+\mathrm{i}p,$$
$$\nabla U=\partial U-\mathrm{i}(v+a)U+\mathrm{i}U(v-a),$$
$$\chi=\partial\chi-\mathrm{i}(v+a)\chi+\mathrm{i}\chi(v-a). \quad \text{(F.1)}$$

Up to the order $O(p^4)$, the Lagrangian can be split into $O(p^2)$, $O(p^3)$ and $O(p^4)$ parts:

$$\mathcal{L}=\mathcal{L}_2+\mathcal{L}_3+\mathcal{L}_4, \quad \text{(F.2)}$$

where

$$\mathcal{L}_n=\sum_{i+j+k=n}\mathcal{L}^{(i,j,k)} \quad \text{(F.3)}$$

and (i,j,k) indicates the number of derivatives, χ sources and powers of B_0, respectively. Then for $O(p^2)$ we get

$$\mathcal{L}^{(2,0,0)}=\frac{F_0^2}{4}\langle\nabla_\mu U^+\nabla^\mu U\rangle,$$
$$\mathcal{L}^{(0,1,1)}=\frac{F_0^2}{2}B_0\langle U^+\chi+\chi^+U\rangle,$$
$$\mathcal{L}^{(0,2,0)}=\frac{F_0^2}{4}\left(\overline{A}_0\langle(U^+\chi)^2+(\chi^+U)^2\rangle+\overline{Z}_0^S\langle U^+\chi+\chi^+U\rangle^2+\overline{Z}_0^P\langle U^+\chi-\chi^+U\rangle^2\right). \quad \text{(F.4)}$$

At the order $O(p^3)$ one has

$$
\begin{aligned}
\mathcal{L}^{(2,1,0)} &= \frac{F_0^2}{4}\big(\overline{\xi}\langle\nabla_\mu U^+\nabla^\mu U(\chi^+U+U^+\chi)\rangle \\
&\quad +\overline{\tilde{\xi}}\langle\nabla_\mu U^+\nabla^\mu U\rangle\langle\chi^+U+U^+\chi\rangle\big)\,, \\
\mathcal{L}^{(0,3,0)} &= \frac{F_0^2}{4}\big(\overline{\rho}_1\langle(\chi^+U)^3+(U^+\chi)^3\rangle \\
&\quad +\overline{\rho}_2\langle(\chi^+U+U^+\chi)\chi^+\chi\rangle \\
&\quad +\overline{\rho}_3\langle(\chi^+U)^2-(U^+\chi)^2\rangle\langle\chi^+U-U^+\chi\rangle \\
&\quad +\overline{\rho}_4\langle(\chi^+U)^2+(U^+\chi)^2\rangle\langle\chi^+U+U^+\chi \\
&\quad +\overline{\rho}_5\langle\chi^+U+U^+\chi\rangle\langle\chi^+\chi\rangle \\
&\quad +\overline{\rho}_6\langle\chi^+U-U^+\chi\rangle^2\langle\chi^+U+U^+\chi\rangle \\
&\quad +\overline{\rho}_7\langle\chi^+U+U^+\chi\rangle^3\big)\,, \\
\mathcal{L}^{(2,0,1)} &= \frac{F_0^2B_0}{4}\delta_d^{(1)}\langle\nabla_\mu U^+\nabla^\mu U\rangle\,, \\
\mathcal{L}^{(0,2,1)} &= \frac{F_0^2B_0}{4}\Big(\delta^{(1)}\overline{A}_0\langle(U^+\chi)^2+(\chi^+U)^2\delta^{(1)}\overline{Z}_0^S \\
&\quad +\langle U^+\chi+\chi^+U\rangle^2+\delta^{(1)}\overline{Z}_0^P\langle U^+\chi-\chi^+U\rangle^2\Big)\,, \\
\mathcal{L}^{(0,1,2)} &= \frac{F_0^2}{2}B_0^2\delta_\chi^{(1)}\langle U^+\chi+\chi^+U\rangle\,.
\end{aligned}
\tag{F.5}
$$

For the $O(p^4)$ Lagrangian, the building blocks are

$$
\begin{aligned}
\mathcal{L}^{(4,0,0)} &= L_1\langle\nabla_\mu U^+\nabla^\mu U\rangle^2+L_2\langle\nabla_\mu U^+\nabla_\nu U\rangle\langle\nabla^\mu U^+\nabla^\nu U\rangle \\
&\quad +L_3\langle\nabla_\mu U^+\nabla^\mu U\nabla_\nu U^+\nabla^\nu U\rangle \\
&\quad -iL_9\langle F^{\mathrm{R}}_{\mu\nu}\nabla^\mu U\nabla^\nu U^+ + F^L_{\mu\nu}\nabla^\mu U^+\nabla^\nu U\rangle \\
&\quad +L_{10}\langle U^+F^{\mathrm{R}}_{\mu\nu}UF^L_{\mu\nu}\rangle \\
&\quad +H_1\langle F^{\mathrm{R}}_{\mu\nu}F^{R\mu\nu}F^L_{\alpha\beta}F^{L\alpha\beta}\rangle\,, \\
\mathcal{L}^{(2,1,1)} &= \frac{F_0^2B_0}{4}\big(\delta^{(1)}\overline{\xi}\langle\nabla_\mu U^+\nabla^\mu U(\chi^+U+U^+\chi)\rangle \\
&\quad +\delta^{(1)}\overline{\tilde{\xi}}\langle\nabla_\mu U^+\nabla^\mu U\rangle\langle\chi^+U+U^+\chi\rangle\big)\,, \\
\mathcal{L}^{(2,0,2)} &= \frac{F_0^2B_0^2}{4}\delta_d^{(2)}\langle\nabla_\mu U^+\nabla^\mu U\rangle \\
\mathcal{L}^{(2,2,0)} &= \frac{F_0^2}{4}\Big\{A_1\langle\nabla_\mu U^+\nabla^\mu U(\chi^+\chi+U^+\chi\chi^+U)\rangle\,, \\
&\quad +A_2\langle(\nabla_\mu U^+)U\chi^+(\nabla^\mu U)U^+\chi\rangle \\
&\quad +A_3\langle\nabla_\mu U^+U(\chi^+\nabla^\mu\chi-\nabla^\mu\chi^+\chi) \\
&\quad +\nabla_\mu UU^+(\chi\nabla^\mu\chi^+-\nabla^\mu\chi\chi^+)\rangle \\
&\quad +A_4\langle\nabla_\mu U^+\nabla^\mu U\rangle\langle\chi^+\chi\rangle \\
&\quad +B_1\langle\nabla_\mu U^+\nabla^\mu U(\chi^+U\chi^+U+U^+\chi U^+\chi)\rangle \\
&\quad +B_2\langle\nabla_\mu U^+\chi\nabla^\mu U^+\chi+\chi^+\nabla^\mu U\chi^+\nabla_\mu U\rangle \\
&\quad +B_4\langle\nabla_\mu U^+\nabla^\mu U\rangle\langle\chi^+U\chi^+U+U^+\chi U^+\chi\rangle \\
&\quad +C_1^S\langle\nabla_\mu U\chi^++\chi\nabla_\mu U^+\rangle\langle\nabla^\mu U\chi^++\chi\nabla^\mu U^+\rangle \\
&\quad +C_2^S\langle\nabla_\mu\chi^+U+U^+\nabla_\mu\chi\rangle\langle\nabla^\mu\chi^+U+U^+\nabla^\mu\chi\rangle \\
&\quad +C_3^S\langle\nabla_\mu\chi^+U+U^+\nabla_\mu\chi\rangle\langle\nabla^\mu U^+\chi+\chi^+\nabla^\mu U\rangle \\
&\quad +C_1^P\langle\nabla_\mu U\chi^+-\chi\nabla_\mu U^+\rangle\langle\nabla^\mu U\chi^+-\chi\nabla^\mu U^+\rangle \\
&\quad +C_2^P\langle\nabla_\mu\chi^+U-U^+\nabla_\mu\chi\rangle\langle\nabla^\mu\chi^+U-U^+\nabla^\mu\chi\rangle \\
&\quad +C_3^P\langle\nabla_\mu\chi^+U-U^+\nabla_\mu\chi\rangle\langle\nabla^\mu U^+\chi-\chi^+\nabla^\mu U\rangle \\
&\quad +D^S\langle\nabla_\mu U^+\nabla^\mu U(\chi^+U+U^+\chi)\rangle\langle\chi^+U+U^+\chi\rangle \\
&\quad +D^P\langle\nabla_\mu U^+\nabla^\mu U(\chi^+U-U^+\chi)\rangle\langle\chi^+U-U^+\chi\rangle\Big\} \\
&\quad +H_2\langle\nabla_\mu\chi\nabla^\mu\chi^+\rangle\,, \\
\mathcal{L}^{(0,4,0)} &= \frac{F_0^2}{4}\{E_1\langle(\chi^+U)^4+(U^+\chi)^4\rangle \\
&\quad +E_2\langle\chi^+\chi(\chi^+U\chi^+U+U^+\chi U^+\chi)\rangle \\
&\quad +E_3\langle\chi^+\chi U^+\chi\chi^+U\rangle \\
&\quad +F_1^S\langle\chi^+U\chi^+U+U^+\chi U^+\chi\rangle^2 \\
&\quad +F_2^S\langle(\chi^+U)^3+(U^+\chi)^3\rangle\langle\chi^+U+U^+\chi\rangle \\
&\quad +F_3^S\langle\chi^+\chi(\chi^+U+U^+\chi)\rangle\langle\chi^+U+U^+\chi\rangle \\
&\quad +F_4^S\langle(\chi^+U)^2+(U^+\chi)^2\rangle\langle\chi^+\chi\rangle \\
&\quad +F_1^P\langle\chi^+U\chi^+U-U^+\chi U^+\chi\rangle^2 \\
&\quad +F_2^P\langle(\chi^+U)^3+(U^+\chi)^3\rangle\langle\chi^+U+U^+\chi\rangle \\
&\quad +F_3^P\langle\chi^+\chi(\chi^+U-U^+\chi)\rangle\langle\chi^+U-U^+\chi\rangle \\
&\quad +F_5^{SS}\langle(\chi^+U)^2+(U^+\chi)^2\rangle\langle\chi^+U+U^+\chi\rangle^2 \\
&\quad +F_6^{SS}\langle\chi^+\chi\rangle\langle\chi^+U+U^+\chi\rangle^2 \\
&\quad +F_5^{SP}\langle(\chi^+U)^2+(U^+\chi)^2\rangle\langle\chi^+U-U^+\chi\rangle \\
&\quad +F_6^{SP}\langle\chi^+\chi\rangle\langle\chi^+U-U^+\chi\rangle^2 \\
&\quad +F_7^{SP}\langle(\chi^+U)^2-(U^+\chi)^2\rangle \\
&\quad \times\langle\chi^+U-U^+\chi\rangle\langle\chi^+U+U^+\chi\rangle \\
&\quad +H_3\langle\chi\chi^+\chi\chi^+\rangle+H_4\langle\chi\chi^+\rangle^2\,, \\
\mathcal{L}^{(0,3,1)} &= \frac{F_0^2B_0}{4}\big(\delta^{(1)}\overline{\rho}_1\langle(\chi^+U)^3+(U^+\chi)^3\rangle \\
&\quad +\delta^{(1)}\overline{\rho}_2\langle(\chi^+U+U^+\chi)\chi^+\chi\rangle \\
&\quad +\delta^{(1)}\overline{\rho}_3\langle(\chi^+U)^2-(U^+\chi)^2\rangle\langle\chi^+U-U^+\chi\rangle \\
&\quad +\delta^{(1)}\overline{\rho}_4\langle(\chi^+U)^2+(U^+\chi)^2\rangle\langle\chi^+U+U^+\chi \\
&\quad +\delta^{(1)}\overline{\rho}_5\langle\chi^+U+U^+\chi\rangle\langle\chi^+\chi\rangle \\
&\quad +\delta^{(1)}\overline{\rho}_6\langle\chi^+U-U^+\chi\rangle^2\langle\chi^+U+U^+\chi\rangle \\
&\quad +\delta^{(1)}\overline{\rho}_7\langle\chi^+U+U^+\chi\rangle^3\big)\,, \\
\mathcal{L}^{(0,2,2)} &= \frac{F_0^2B_0^2}{4}\big(\delta^{(2)}\overline{A}_0\langle(U^+\chi)^2+(\chi^+U)^2\rangle \\
&\quad +\delta^{(2)}\overline{Z}_0^S\langle U^+\chi+\chi^+U\rangle^2 \\
&\quad +\delta^{(2)}\overline{Z}_0^P\langle U^+\chi-\chi^+U\rangle^2\big)\,, \\
\mathcal{L}^{(0,1,3)} &= \frac{F_0^2}{2}B_0^3\delta_\chi^{(2)}\langle U^+\chi+\chi^+U\rangle\,.
\end{aligned}
\tag{F.6}
$$

In fact, identifying F_0 with the Goldstone boson decay constant and $B_0=\Sigma/F_0^2$ where $\Sigma=-\langle\overline{u}u\rangle_0$ (in the chiral limit), we have

$$
\delta_d^{(i)}=\delta_\chi^{(i)}=0\,. \tag{F.7}
$$

As usual, we can also resum the powers of B_0 already at the Lagrangian level and write

$$
\mathcal{L}_n=\sum_{i+j=n}\mathcal{L}^{(i,j)}\,, \tag{F.8}
$$

with

$$
\mathcal{L}^{(i,j)}=\sum_k\mathcal{L}^{(i,j,k)} \tag{F.9}
$$

and denote

$$\begin{aligned}
A_0 &= \overline{A}_0 + B_0\delta^{(1)}\overline{A}_0 + B_0^2\delta^{(2)}\overline{A}_0 + \dots\,, \\
Z_0^{S,P} &= \overline{Z}_0^{S,P} + B_0\delta^{(1)}\overline{Z}_0^{S,P} + B_0^2\delta^{(2)}\overline{Z}_0^{S,P} + \dots\,, \\
\xi &= \overline{\xi} + B_0\delta^{(1)}\overline{\xi} + \dots\,, \\
\tilde{\xi} &= \overline{\tilde{\xi}} + B_0\delta^{(1)}\overline{\tilde{\xi}} + \dots \\
\rho_i &= \overline{\rho}_i + B_0\delta^{(1)}\overline{\rho}_i + \dots\,,
\end{aligned} \tag{F.10}$$

etc. These LECs, without the bars, are used in the main text. Note that while the $O(p^2)$ parameters $\overline{A}_0$, $\overline{Z}_0^{S,P}$ and the $O(p^3)$ LECs $\overline{\tilde{\xi}}$ and $\overline{\xi}$ are renormalization scale independent, the renormalized resumed parameters $Z_0^{S,P,r}$, A_0^r and $\tilde{\xi}^r$, ξ^r run with μ in the same way as $16(B_0/F_0)^2L_{6-8}^r$ and $8B_0/F_0^2L_{4,5}$ within standard χPT.

Appendix G: Coefficients of the dispersive part of the GχPT amplitude

In these formulae as well as in the following two appendices, the masses $\tilde{M}_P^2$ are the generalized $O(p^2)$ masses given by (112). We have

$$\begin{aligned}
\alpha_{\pi\eta}\tilde{M}_\pi^2 &= 2\big[\widehat{m}B_0 + 8\widehat{m}^2A_0 + 2\widehat{m}^2Z_0^S(5r+4) \\
&\quad - 8\widehat{m}^2Z_0^P(r-1)\big]\,, \\
\alpha_{\pi\eta K}\tilde{M}_\pi^2 &= 4\widehat{m}^2(r^2-1)\left(A_0+2Z_0^P\right)\,, \\
\alpha_{\pi\pi}\tilde{M}_\pi^2 &= 2\widehat{m}B_0 + 16\widehat{m}^2A_0 + 4\widehat{m}^2Z_0^S(r+8)\,, \\
\alpha_{\eta\eta}(4\tilde{M}_\eta^2-\tilde{M}_\pi^2) &= \frac{2}{3}\widehat{m}B_0(1+8r) + \frac{16}{3}\widehat{m}^2A_0(1+8r^2) \\
&\quad + \frac{4}{3}\widehat{m}^2Z_0^S(8+41r+32r^2) \\
&\quad + \frac{32}{3}\widehat{m}^2Z_0^P(r-1)(4r-1)\,, \\
(\alpha_{\pi K}-1)\tilde{M}_K\tilde{M}_\pi &= 6(A_0+2Z_0^S)\widehat{m}^2(r+1)\,, \\
\alpha_{\eta K}(2\tilde{M}_\eta^2 - \frac{2}{3}\tilde{M}_K^2) &= \frac{2}{3}\big[\widehat{m}B_0(1+3r) \\
&\quad + \widehat{m}^2A_0(3+10r+19r^2)\,, \\
&\quad + 2\widehat{m}^2Z_0^S(6+19r+11r^2) \\
&\quad + 16\widehat{m}^2Z_0^Pr(r-1)\big]\,.
\end{aligned} \tag{G.1}$$

Appendix H: Parameters α–ω within the generalized χPT

Here we summarize the formulae in terms of the decomposition of the remainders. For the parameter α we write

$$\delta_\alpha = \delta_\alpha^{\rm loop} + 3\frac{\widehat{m}^2F_0^2}{F_\pi^2M_\pi^2}\delta_\alpha^{\rm CT}(\mu) + \delta_\alpha^{{\rm G}\chi{\rm PT}}\,. \tag{H.1}$$

For the counterterm contribution we get

$$\begin{aligned}
\delta_\alpha^{\rm CT}(\mu) &= \frac{1}{3}\widehat{m}\big[81\rho_1 + \rho_2 + (80-64r-16r^2)\rho_3 \\
&\quad + (100+64r+34r^2)\rho_4 + (2+r^2)\rho_5 \\
&\quad + (96-96r)\rho_6 + (144+288r+108r^2)\rho_7\big] \\
&\quad + \frac{8}{3}\Big[-(B_1-B_2)\Sigma_{\pi\eta} + 2D^PM_\pi^2(r-1) \\
&\quad - 2C_1^PM_\eta^2(r-1) - \frac{1}{2}D^S[\Sigma_{\pi\eta}(5r+4)] \\
&\quad - 2B_4\left[3M_\eta^2 + M_\pi^2(2r^2+1)\right]\Big] \\
&\quad + \frac{1}{3}\widehat{m}^2\big[256E_1 + 16E_2 + F_1^P(256-256r^2) \\
&\quad + F_4^S(32+16r^2) + F_1^S(256+320r^2) \\
&\quad + F_5^{SP}(192-320r+160r^2-32r^3) \\
&\quad + F_2^P(240-216r-24r^3) \\
&\quad + F_6^{SP}(32-32r+16r^2-16r^3) \\
&\quad + F_3^P(16-8r-8r^3) + F_3^S(16+10r+10r^3) \\
&\quad + F_6^{SS}(32+40r+16r^2+20r^3) \\
&\quad + F_7^{SP}(384-160r-256r^2+32r^3) \\
&\quad + F_2^S(400+234r+74r^3) \\
&\quad + F_5^{SS}(576+720r+480r^2+168r^3)\big]
\end{aligned} \tag{H.2}$$

and the loops contribute as follows:

$$\begin{aligned}
\frac{1}{3}F_\pi^2M_\pi^2\delta_\alpha^{\rm loop} &= \frac{1}{3}\Big\{\big[\tilde{M}_\pi^2\big(3B_0\widehat{m} + 64A_0\widehat{m}^2 \\
&\quad + 2Z_0^S\widehat{m}^2(15r+32) - 8Z_0^P\widehat{m}^2(3r-8)\big)\big] \\
&\quad - 6B_0^2\widehat{m}^2\Big\}\left(J_{\pi\pi}^r(0) + \frac{1}{16\pi^2}\right) \\
&\quad + \frac{2}{3}\Big\{\big[\tilde{M}_K^2\big(B_0\widehat{m} + 2A_0\widehat{m}^2(r+8) \\
&\quad + 2Z_0^S\widehat{m}^2(15r+8) - 8Z_0^P\widehat{m}^2(3r-2)\big)\big] \\
&\quad - B_0^2\widehat{m}^2(r+1)\Big\}\left(J_{KK}^r(0) + \frac{1}{16\pi^2}\right) \\
&\quad \times\frac{1}{9}\Big\{\big[\tilde{M}_\eta^2\big(B_0\widehat{m} + 32A_0\widehat{m}^2 \\
&\quad - 16Z_0^P\widehat{m}^2(5r-2) + 2Z_0^S\widehat{m}^2(41r+16)\big)\big] \\
&\quad - \frac{2}{3}B_0^2\widehat{m}^2(2r+1)\Big\}\left(J_{\eta\eta}^r(0) + \frac{1}{16\pi^2}\right) \\
&\quad + \frac{2}{9}\Big\{\big[\tilde{M}_\pi^2 + 4\widehat{m}^2\big(3A_0 - 4(r-1)Z_0^P \\
&\quad + 2(2r+1)Z_0^S\big)\big]^2 - 4B_0^2\widehat{m}^2\Big\}J_{\pi\eta}^r(0) \\
&\quad + \frac{3}{4}\Big\{\left[\frac{2}{3}\tilde{M}_\pi^2 - \frac{8}{3}(r-1)\widehat{m}^2\left(A_0+2Z_0^P\right)\right]^2 \\
&\quad - \frac{16}{9}B_0^2\widehat{m}^2\Big\}J_{KK}^r(0) \\
&\quad + \frac{1}{3}\Big\{\big[\tilde{M}_\pi^2 + 4\widehat{m}^2\big(3A_0 - 4(r-1)Z_0^P \\
&\quad + 2(2r+1)Z_0^S\big)\big]
\end{aligned}$$

$$\times\left[-2M_\pi^2+\frac{3}{2}\tilde{M}_\pi^2+10\widehat{m}^2\left(A_0+2Z_0^S\right)\right]$$
$$-2B_0\widehat{m}\left(3B_0\widehat{m}-2M_\pi^2\right)\Big\}J_{\pi\pi}^r(0)$$
$$+\frac{2}{9}\Big\{\left[\tilde{M}_\pi^2+4\widehat{m}^2\big(3A_0-4(r-1)Z_0^P\right.$$
$$\left.+2(2r+1)Z_0^S\big)\right]\left[\tilde{M}_\eta^2-\frac{1}{4}\tilde{M}_\pi^2\right.$$
$$\times+\widehat{m}^2\big((8r^2+1)A_0+8r(r-1)Z_0^P+2$$
$$\left.\times(2r+1)^2Z_0^S\big)\right]-\frac{1}{3}B_0^2\widehat{m}^2(8r+1)\Big\}J_{\eta\eta}^r(0)$$
$$+\frac{1}{8}\Big\{\big[-2M_\pi^2+2\tilde{M}_\pi^2+8(r+1)\widehat{m}^2$$
$$\times\left(A_0+2Z_0^S\right)\big]\left[-6M_\eta^2+6\tilde{M}_\eta^2-\frac{8}{3}\tilde{M}_K^2\right.$$
$$+\frac{8}{3}(r+1)\widehat{m}^2\big(3rA_0+2(r-1)Z_0^P$$
$$\left.+2(2r+1)Z_0^S\big)\right]-2\left(2B_0\widehat{m}-M_\pi^2\right)$$
$$\times\left(\frac{4}{3}B_0\widehat{m}(4r+1)-6M_\eta^2\right)\Big\}J_{KK}^r(0)\,. \tag{H.3}$$

In the same way we have for β

$$\beta\delta_\beta=\beta\delta_\beta^{\mathrm{loop}}+\widehat{m}^2F_0^2\delta_\beta^{\mathrm{CT}}(\mu)+\beta\delta_\beta^{\mathrm{G}\chi\mathrm{PT}}\,, \tag{H.4}$$

where

$$\delta_\beta^{\mathrm{CT}}(\mu)=\frac{8}{3}[(C_1^S+D^S)(2r+1)+2B_4(r^2+2)] \tag{H.5}$$
$$\beta\delta_\beta^{\mathrm{loop}}=-\frac{3}{4}\Big\{\left[\frac{2}{3}\tilde{M}_\pi^2-\frac{8}{3}(r-1)\widehat{m}^2\left(A_0+2Z_0^P\right)\right]$$
$$-\frac{4}{3}B_0\widehat{m}\Big\}J_{KK}^r(0)+\frac{1}{3}\Big\{\big[\tilde{M}_\pi^2+4\widehat{m}^2$$
$$\times\big(3A_0-4(r-1)Z_0^P+2(2r+1)Z_0^S\big)\big]-2B_0\widehat{m}\Big\}$$
$$\times J_{\pi\pi}^r(0)+\frac{1}{8}\Big\{\left[6(\tilde{M}_\eta^2-M_\eta^2+\tilde{M}_\pi^2-M_\pi^2)-\frac{8}{3}\tilde{M}_K^2\right.$$
$$+\frac{8}{3}(r+1)\widehat{m}^2\big(3A_0(r+3)+4Z_0^S(r+5)$$
$$\left.+2(r-1)Z_0^P\big)\right]$$
$$-\left[\frac{8}{3}B_0\widehat{m}(2r+5)-6M_\eta^2-6M_\pi^2\right]\Big\}J_{KK}^r(0)\,. \tag{H.6}$$

For the remaining two parameters the corresponding decomposition of the remainders,

$$\gamma\delta_\beta(\mu)=\gamma\delta_\gamma{}^{\mathrm{loops}}(\mu)+\widehat{m}^2F_0^2\delta_\gamma^{\mathrm{CT}}(\mu)+\gamma\delta_\gamma^{\mathrm{G}\chi\mathrm{PT}}\,, \tag{H.7}$$
$$\omega\delta_\omega(\mu)=\omega\delta_\omega{}^{\mathrm{loops}}(\mu)+\widehat{m}^2F_0^2\delta_\omega^{\mathrm{CT}}(\mu)+\omega\delta_\omega^{\mathrm{G}\chi\mathrm{PT}}\,, \tag{H.8}$$

is trivial, i.e.

$$\delta_\gamma{}^{\mathrm{loops}}(\mu)=\delta_\gamma^{\mathrm{CT}}(\mu)=\delta_\omega{}^{\mathrm{loops}}(\mu)=\delta_\omega^{\mathrm{CT}}(\mu)=0\,. \tag{H.9}$$

Appendix I: Generalized χPT contributions to the bare expansion remainders for the masses

The expressions for $\xi,\tilde{\xi}$ can be obtained from the exact algebraic identities (E.2) after the identification (117) and using the representation (B.10) and (B.11) for the remainder of F_π^2 and (B.12) and (B.13) for the remainder of F_K^2. In the same spirit, A_0, Z_0^S and Z_0^P can be expressed using the identities (E.5) and the following remainders:

$$F_\pi^2M_\pi^2\delta_{F_\pi M_\pi}=F_\pi^2M_\pi^2\delta_{F_\pi M_\pi}^{\mathrm{loop}}(\mu)+F_0^2\widehat{m}^2\delta_{F_\pi M_\pi}^{\mathrm{CT}}(\mu)+F_\pi^2M_\pi^2\delta_{F_\pi M_\pi}^{\mathrm{G}\chi\mathrm{PT}}\,,$$
$$F_K^2M_K^2\delta_{F_K M_K}=F_K^2M_K^2\delta_{F_K M_K}^{\mathrm{loop}}(\mu)+F_0^2\widehat{m}^2\delta_{F_K M_K}^{\mathrm{CT}}(\mu)+F_K^2M_K^2\delta_{F_K M_K}^{\mathrm{G}\chi\mathrm{PT}}\,,$$
$$F_\eta^2M_\eta^2\delta_{F_\eta M_\eta}=F_\eta^2M_\eta^2\delta_{F_\eta M_\eta}^{\mathrm{loop}}(\mu)+F_0^2\widehat{m}^2\delta_{F_\eta M_\eta}^{\mathrm{CT}}(\mu)+F_\eta^2M_\eta^2\delta_{F_\eta M_\eta}^{\mathrm{G}\chi\mathrm{PT}}\,, \tag{I.1}$$

where

$$F_\pi^2M_\pi^2\delta_{F_\pi M_\pi}^{\mathrm{loop}}(\mu)=$$
$$\left[\tilde{M}_\pi^2\left(3B_0\widehat{m}+16A_0\widehat{m}^2+2Z_0^S\widehat{m}^2(3r+16)\right)-6B_0^2\widehat{m}^2\right]$$
$$\times\left(J_{\pi\pi}^r(0)+\frac{1}{16\pi^2}\right)$$
$$+2\left[\tilde{M}_K^2\left(B_0\widehat{m}+2A_0\widehat{m}^2(r+2)+2Z_0^S\widehat{m}^2(3r+4)\right)\right.$$
$$\left.-B_0^2\widehat{m}^2(r+1)\right]\left(J_{KK}^r(0)+\frac{1}{16\pi^2}\right)$$
$$+\frac{1}{3}\left[\tilde{M}_\eta^2\big(B_0\widehat{m}+8A_0\widehat{m}^2-8Z_0^P\widehat{m}^2(r-1)+2Z_0^S\widehat{m}^2\right.$$
$$\left.\times(5r+4)-\frac{2}{3}B_0^2\widehat{m}^2(2r+1)\big)\right]\left(J_{\eta\eta}^r(0)+\frac{1}{16\pi^2}\right), \tag{I.2}$$

$$\delta_{F_\pi M_\pi}^{\mathrm{CT}}(\mu)=\widehat{m}\big[9\rho_1+\rho_2+2\rho_4(10+4r+r^2)$$
$$+\rho_5(2+r^2)+12\rho_7(4+4r+r^2)\big]$$
$$+2\widehat{m}^2\big[8E_1+2E_2+8F_1^S(2+r^2)$$
$$+F_2^S(9r+r^3+20)+F_3^S(4+r+r^3)$$
$$+2F_4^S(2+r^2)+4F_5^{SS}(r+2)(r^2+2r+6)$$
$$+2F_6^{SS}(r+2)(2+r^2)\big]\,; \tag{I.3}$$

then

$$F_K^2M_K^2\delta_{F_K M_K}^{\mathrm{loop}}(\mu)=$$
$$\Big\{\frac{3}{4}\left[\tilde{M}_\pi^2\left(B_0\widehat{m}+A_0\widehat{m}^2(r+5)+2Z_0^S\widehat{m}^2(r+6)\right)-2B_0^2\widehat{m}^2\right]$$

$$\times\left(J^r_{\pi\pi}(0)+\frac{1}{16\pi^2}\right)+\frac{3}{2}\big[\tilde{M}_K^2\big(B_0\widehat{m}+3A_0\widehat{m}^2(r+1) +2Z_0^S\widehat{m}^2(3r+4)\big)-B_0^2\widehat{m}^2(r+1)\big] \times\left(J^r_{KK}(0)+\frac{1}{16\pi^2}\right) +\frac{1}{12}\Big[\tilde{M}_\eta^2\big(5B_0\widehat{m}+A_0\widehat{m}^2(17r+5)+8Z_0^P\widehat{m}^2(r-1) +2Z_0^S\widehat{m}^2(13r+14)\big)-\frac{10}{3}B_0^2\widehat{m}^2(2r+1)\Big] \times\left(J^r_{\eta\eta}(0)+\frac{1}{16\pi^2}\right)\Bigg\}(r+1)\,, \tag{I.4}$$

$$\delta^{\mathrm{CT}}_{F_KM_K}(\mu)=\frac{1}{2}\widehat{m}\big[3\rho_1(1+r)(1+r+r^2)+\rho_2(1+r^3) +6\rho_4(r+1)(2+2r+r^2)+\rho_5(r+1)(2+r^2) +12\rho_7(r+1)(r+2)^2\big] +\widehat{m}^2\,\big[2E_1(1+r)^2(1+r^2) +E_2(1+r)^2(1-r+r^2)+\frac{1}{2}E_3\,(r^2-1)^2 +4F_1^S(1+r)^2(2+r^2) +F_2^S(1+r)(8+9r+9r^2+4r^3) -F_3^S(1+r)(4-r+r^2+2r^3) +F_4^S(1+r)^2(2+r^2) +4F_5^{SS}(1+r)(2+r)(4+3r+2r^2) +2F_6^{SS}(1+r)(2+r)(2+r^2)\big] \tag{I.5}$$

and

$$F_\eta^2M_\eta^2\delta^{\mathrm{loop}}_{F_\eta M_\eta}(\mu)=\big[\tilde{M}_\pi^2\big(B_0\widehat{m}+8A_0\widehat{m}^2-8\widehat{m}^2Z_0^P(r-1) +2\widehat{m}^2Z_0^S(4+5r)\big)-2B_0^2\widehat{m}^2\big]\left(J^r_{\pi\pi}(0)+\frac{1}{16\pi^2}\right) +\frac{2}{3}\big[\tilde{M}_K^2\big(B_0\widehat{m}\,(1+4r)+2A_0\widehat{m}^2\,(2+r+8r^2) +8\widehat{m}^2Z_0^P(r-1)(2r-1)+2\widehat{m}^2Z_0^S(4+15r+8r^2)\big) -B_0^2\widehat{m}^2(4r+1)(r+1)\big]\left(J^r_{KK}(0)+\frac{1}{16\pi^2}\right) +\frac{1}{9}\Big[\tilde{M}_\eta^2\big(B_0\widehat{m}(1+8r)+8A_0\widehat{m}^2(1+8r^2) +16\widehat{m}^2Z_0^P(r-1)(4r-1)+2\widehat{m}^2Z_0^S(8+41r+32r^2)\big) -\frac{2}{3}B_0^2\widehat{m}^2(8r+1)(2r+1)\Big]\left(J^r_{\eta\eta}(0)+\frac{1}{16\pi^2}\right), \tag{I.6}$$

$$\delta^{\mathrm{CT}}_{F_\eta M_\eta}(\mu)=\frac{1}{3}\widehat{m}\big[9\rho_1(1+2r^3)+\rho_2(1+2r^3) +16\rho_3(r-1)^2(1+r) +2\rho_4(10+8r+17r^2+10r^3) +\rho_5(1+2r)(2+r^2)+16\rho_6(2-3r+r^3) +12\rho_7(2+r)^2(1+2r)\big] +\frac{2}{3}\widehat{m}^2\big[8E_1(1+2r^4)+2E_2(1+2r^4) +8F_1^S(r^2+2)(2r^2+1)+16F_1^P(r^2-1)^2 +F_2^S(20+13r+37r^3+20r^4) +12F_2^P(r^2+r+1)(r-1)^2 +F_3^S(4+5r+5r^3+4r^4) +4F_3^P(r^2+r+1)(r-1)^2 +2F_4^S(r^2+2)(2r^2+1) +12F_5^{SS}(r+2)(2r^3+3r^2+2r+2) +8F_5^{SP}(r^2+2)(r-1)^2 +2F_6^{SS}(2r+1)(r+2)(r^2+2) +4F_6^{SP}(r^2+2)(r-1)^2 +16F_7^{SP}(r+2)(r+1)(r-1)^2\big]\,. \tag{I.7}$$

Appendix J: Resonance amplitude and remainders estimates

Here we give the contribution to the amplitude [38] related to the resonance exchange, derived from the leading order Lagrangian of RχT (we have confirmed this expression by independent calculation)

$$G_{\mathrm{R}}(s,t;u)=4\frac{1}{M_{S_1}^2-t}\left(\tilde{c}_d\left(t-2M_\pi^2\right)+2\tilde{c}_m\,\overset{o}{M}{}_\pi^2\right) \times\left(\tilde{c}_d\left(t-2M_\eta^2\right)+2\tilde{c}_m\,\overset{o}{M}{}_\eta^2\right) +4\frac{\tilde{c}_m^2}{M_{S_1}^2}\,\overset{o}{M}{}_\pi^2\left(\overset{o}{M}{}_\pi^2+\overset{o}{M}{}_\eta^2\right) +4\frac{c_m^2}{3M_S^2}\,\overset{o}{M}{}_\pi^2\left(\overset{o}{M}{}_\pi^2-\overset{o}{M}{}_\eta^2\right) +\frac{2}{3}\frac{1}{M_S^2-s}\left(c_d\left(s-M_\pi^2-M_\eta^2\right)+2c_m\,\overset{o}{M}{}_\pi^2\right)^2 +\frac{2}{3}\frac{1}{M_S^2-u}\left(c_d\left(u-M_\pi^2-M_\eta^2\right)+2c_m\,\overset{o}{M}{}_\pi^2\right)^2 -\frac{2}{3}\frac{1}{M_S^2-t}\left(c_d\left(t-2M_\pi^2\right)+2c_m\,\overset{o}{M}{}_\pi^2\right) \times\left(c_d\left(t-2M_\eta^2\right)+2c_m\left(2\,\overset{o}{M}{}_\eta^2-\overset{o}{M}{}_\pi^2\right)\right) -16\frac{\tilde{d}_m^2}{M_{\eta_1}^2}\,\overset{o}{M}{}_\pi^2\left(\overset{o}{M}{}_\pi^2-\overset{o}{M}{}_\eta^2\right). \tag{J.1}$$

The resonance estimate of the remainders $\delta_\gamma^{\mathrm{R}}$ and $\delta_\omega^{\mathrm{R}}$ are

$$\gamma\delta_\gamma^{\mathrm{R}}=-\frac{8}{3M_S^6}\left(c_dM_\pi^2-c_m\,\overset{o}{M}{}_\pi^2\right) \times\left(c_dM_\eta^2-c_m\left(2\,\overset{o}{M}{}_\eta^2-\overset{o}{M}{}_\pi^2\right)\right) +\frac{4}{3}\frac{c_d}{M_S^4}\left(c_dM_\eta^2-c_m\left(2\,\overset{o}{M}{}_\eta^2-\overset{o}{M}{}_\pi^2\right)\right) +\frac{1}{3}\frac{c_d^2\Sigma_{\pi\eta}}{M_S^2\left(M_S^2-\Sigma_{\pi\eta}\right)}$$

$$+\frac{4}{3}\frac{c_d c_m}{(M_S^2-\Sigma_{\pi\eta})^2}\overset{o}{M}{}_\pi^2+\frac{4}{3}\frac{c_m^2}{(M_S^2-\Sigma_{\pi\eta})^3}\overset{o}{M}{}_\pi^4$$
$$+\frac{16}{M_{S_1}^6}\left(\tilde{c}_d M_\pi^2-\tilde{c}_m\overset{o}{M}{}_\pi^2\right)\left(\tilde{c}_d M_\eta^2-\tilde{c}_m\overset{o}{M}{}_\eta^2\right)$$
$$-\frac{8\tilde{c}_d}{M_{S_1}^4}\left(\tilde{c}_d\Sigma_{\pi\eta}-\tilde{c}_m\left(\overset{o}{M}{}_\pi^2+\overset{o}{M}{}_\eta^2\right)\right), \tag{J.2}$$

$$\omega\delta_\omega^{\mathrm{R}}=-\frac{1}{3}\frac{c_d^2\Sigma_{\pi\eta}}{M_S^2(M_S^2-\Sigma_{\pi\eta})}+\frac{4}{3}\frac{c_m^2\overset{o}{M}{}_\pi^4}{(M_S^2-\Sigma_{\pi\eta})^3}$$
$$+\frac{4}{3}\frac{c_d c_m}{(M_S^2-\Sigma_{\pi\eta})^2}\overset{o}{M}{}_\pi^2\;. \tag{J.3}$$

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